PROPERTIES OF MEASURES SUPPORTED ON FAT SIERPINSKI CARPETS

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Abstract. In this article we study certain conformal iterated function schemes in two dimensions which are natural generalizations of the Sierpinski carpet construction. In particular, we consider scaling factors for which the open set condition fails. For such 'Fat Sierpinski carpets' we study the range of parameters for which the dimension of the set is exactly known, or for which the set has positive measure.

0. Introduction

In this note we want to study a simple conformal iterated function scheme which fails to satisfy the standard open set condition. Let $0 < \lambda < 1$. Given $n > k$ we want to consider a family of n conformal contractions $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$T_i : (x, y) \mapsto (\lambda x, \lambda y) + (c_i^{(1)}, c_i^{(2)}),$$

$i = 1, \ldots, n$, where $(c_i^{(1)}, c_i^{(2)}) \in \left\{(j, l) \in \mathbb{Z}^2 : 0 \leq j, l \leq k-1\right\}$ are n distinct points in a $k \times k$ grid. There is then a unique smallest closed set Λ_λ such that $\Lambda_\lambda = \cup_{i=1}^k T_i(\Lambda_\lambda)$. In the special case that $\lambda = \frac{1}{k}$, the sets $\Lambda_{\frac{1}{k}}$ are the well-known Sierpinski carpets. If $\lambda \in (0, \frac{1}{k})$ then the contractions satisfy the open set condition and Λ_λ is a Cantor set whose dimension we can easily compute as

$$\dim_H(\Lambda_\lambda) = -\frac{\log n}{\log \lambda}. \tag{0.1}$$

In this note we shall extend this equality to a strictly larger parameter set of λ. Unfortunately, we cannot expect this identity to hold on a larger interval since it is easy to see that there are examples with a countable dense set of exceptional values $\mathcal{E} \subset [\frac{1}{k}, \frac{1}{\sqrt{n}}]$ such that for $\lambda \in \mathcal{E}$ we have (0.1) fails. Our first result extends these results to a larger set.

The diagrams were drawn using Bob Devaney's programme Fractalina and the numerical calculations done using Mathematica. We would like to thank Nikita Sidorov for useful conversations.

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Theorem 1. *There exists* $\frac{1}{k} \leq s \leq \frac{1}{\sqrt{n}}$ *such that for almost all* $\lambda \in (\frac{1}{k}, s]$ *we have* $\dim_H(\Lambda_\lambda)$ *is given by (0.1).*

We can give an explicit estimate for s. More precisely, we denote the number of images in the jth row by $n_j = \mathrm{Card}\left\{1 \leq l \leq k : c_i^{(1)} = j\right\}$, for $j = 1, \ldots, k$. If we Assume that $n_i \geq 1$, then we can take

$$s = \min\left\{\frac{1}{n}\left(\prod_{j=1}^{k} n_j^{n_j}\right)^{\frac{1}{n}}, \left(\prod_{j=1}^{k} n_j^{-n_j}\right)^{\frac{1}{n}}\right\} \tag{0.2}$$

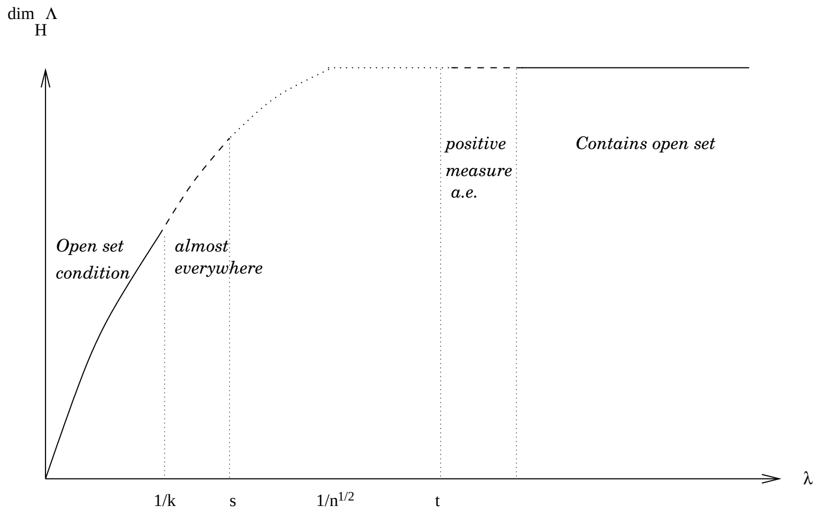

FIGURE 1. EXTENDING THE REGIONS WHERE DIMENSION IS KNOWN

The following is a simple Corollary to Theorem 1.

Corollary. *There exists a dense* G_δ *set* $\mathcal{G} \subset [\frac{1}{k}, s]$ *such that for* $\lambda \in \mathcal{G}$ *we have* $\dim_H(\Lambda_\lambda)$ *is given by (0.1).*

Proof. This follows from the semi-continuity of the map $\lambda \mapsto \dim_H(\Lambda_\lambda)$ [19], [6]. □

Providing λ is sufficiently large, we might expect the set to have positive measure. An ingredient in the study of this problem is a development of the idea transversality [16]. This leads to a technical constraint in proving these theorems which requires that $t \leq b_{k-1} \ldots$, a transversality constant. For example, $b_1 = 0.649\ldots$ and $b_2 = 0.5$.

Theorem 2. *There exists* $\frac{1}{\sqrt{n}} \leq t \leq b_{k-1}$ *such that for almost all* $\lambda \in [t, b_{k-1}]$ *we have that* $\mathrm{leb}(\Lambda_\lambda) > 0$.

We can give an explicit estimate for $t = t(n_1, \ldots, n_k)$:

$$t = \sup\left\{\prod_{j=1}^{k} q_j^{n_j} : \sum_{j=1}^{k} q_j \log\left(\frac{q_j}{n_j}\right) = 0, \sum_{j=1}^{k} q_j = 1 \text{ and } q_j \geq 0\right\}. \tag{0.3}$$

Example 1: The Sierpinski triangle. Let $k = 2$ and $c_1 = (0,0)$, $c_2 = (1,0)$ and $c_3 = (0,1)$. Broomhead, Montaldi and Sidorov computed the dimension of Λ_λ at certain exceptional values $\omega_n \searrow 0$, called multinacci numbers, characterised as roots of $3x^{n+1} - 3x + 1$ (e.g., $\omega_2 = 0.618\ldots$, $\omega_3 = 0.543\ldots$, $\omega_4 = 0.518\ldots$ etc.) [4].

Jordan established Theorem 1 with $s = 2^{\frac{2}{3}}/3 = 0.529\ldots$ [7]. Theorem 2 applies with $t = 0.5852\ldots$ (corresponding to the choice $q_2 = 0.7729\ldots$). For comparison, in [4] it is shown that for $\lambda \geq 0.647\ldots$ the set Λ_λ contains open sets.

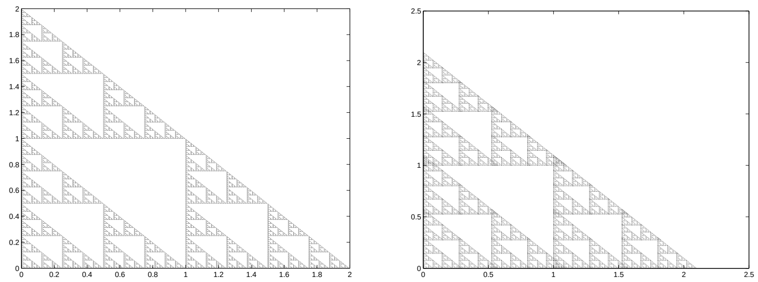

FIGURE 2. SIERPINSKI GASKET WHERE (I) $\lambda = 0.5$ AND (II) $\lambda = 0.525$

Example 2: The Sierpinski carpet. Let $k = 3$ and $c_1, \ldots, c_8$ are all but the central square. In Theorem 1, we can take $s = (3^3 2^3 3^3)^{\frac{1}{8}}/8 = 0.338851\ldots$. In Theorem 2, we can take $t = 0.357\ldots$ (corresponding to the choices $q_1 = q_3 = 0.416\ldots$ and $q_2 = 0.168\ldots$).[1]

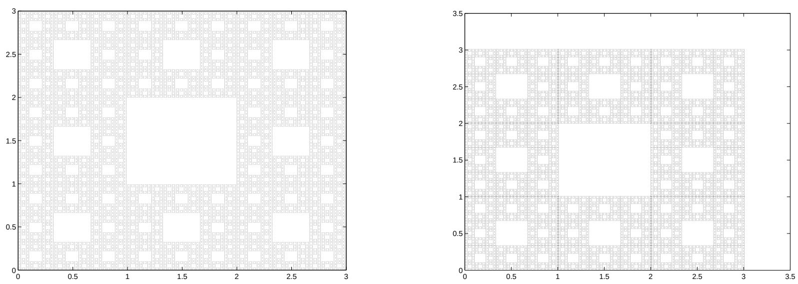

FIGURE 3. SIERPINSKI CARPETS WHERE (I) $\lambda = \frac{1}{3}$ AND (II) $\lambda = 0.338$

Example 3 : Vicsek set. Let $k = 3$ and let $c_1, \ldots, c_5$ correspond to a cross. In this case $s = (3^3)^{\frac{1}{5}}/5 = 0.386636\ldots$. and $t = 0.4541$.

These are summarised in the following table.

Shape	$\frac{1}{k}$	s	$\frac{1}{\sqrt{n}}$	t
Triangle	0.5	$0.529\ldots$	$0.577\ldots$	$0.585\ldots$
Carpet	$0.333\ldots$	$0.338\ldots$	$0.353\ldots$	$0.357\cdots$
Cross	$0.333\ldots$	$0.386\ldots$	$0.447\ldots$	$0.454\ldots$

[1] One trivially sees that one could choose $t = \frac{1}{2}$, since for $\lambda \geq \frac{1}{2}$ we have that Λ_λ is a square, and thus has positive measure.

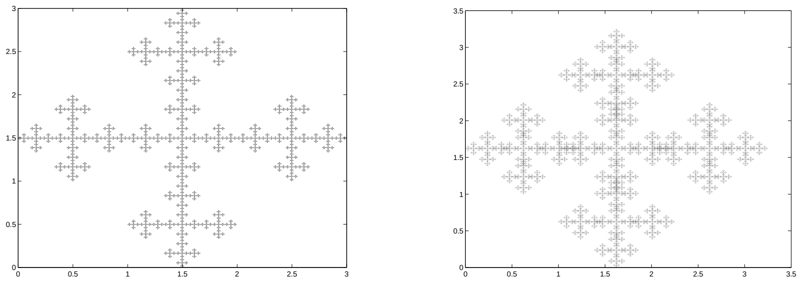

FIGURE 4. VICSEK SETS WHERE (I) $\lambda = \frac{1}{3}$ AND (II) $\lambda = 0.386$

In sections 1 we formulate a general result on projections of measures. In sections 2 and 3 we develop the technical results. The proof of Theorem 1 is completed in section 3. In sections 4 and 5 we give the proof of Theorem 2. In section 6, we consider generalizations.

1. SUBSHIFTS AND INVARIANT MEASURES.

Let $\Sigma_n = \{1, 2, \cdots, n\}^{\mathbb{Z}^+}$ be the space of sequences and let $\sigma : \Sigma_n \to \Sigma_n$ be the full shift on n symbols defined by $(\sigma x)_n = x_{n+1}$. Let $\Pi_\lambda : \Sigma_n \to \mathbb{R}^2$ be defined by

$$\Pi_\lambda(x_m) = \sum_{m=0}^{\infty} c_{x_m} \lambda^m.$$

The (two dimensional) fat Sierpinski carpet $\Lambda_\lambda \subset \mathbb{R}^2$ is defined by

$$\Lambda_\lambda = \left\{ \Pi_\lambda(x) = \sum_{m=0}^{\infty} c_{x_m} \lambda^m \, : \, x = (x_m)_{m=0}^{\infty} \in \Sigma_n \right\}.$$

Let $\Sigma_k = \{1, 2, \ldots, k\}^{\mathbb{Z}^+}$. We can define a factor map $p : \Sigma_n \to \Sigma_k$ by $(p(x))_i = c_{x_i}^{(1)}$ where $c_{x_i} = (c_{x_i}^{(1)}, c_{x_i}^{(2)})$, for $i \in \mathbb{Z}^+$. Let $\overline{\Pi}_\lambda : \Sigma_k \to \mathbb{R}$ be defined by $\overline{\Pi}_\lambda(y_m) = \sum_{m=0}^{\infty} y_m \lambda^m$. We can associate a closed set $\Lambda_\lambda \subset \mathbb{R}$ to Σ_k defined by

$$\overline{\Lambda}_\lambda = \left\{ \overline{\Pi}_\lambda(y) = \sum_{m=0}^{\infty} y_m \lambda^m \, : \, y = (y_m)_{m=0}^{\infty} \in \Sigma_k \right\}.$$

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the horizontal projection $\pi(x, y) = y$ on the vertical axis. Then we can write $\overline{\Pi}_\lambda \circ p = \pi \circ \Pi_\lambda$.

Let μ be an ergodic shift invariant probability measure on Σ_n. The image $\overline{\mu} := p(\mu)$ of μ under $p : \Sigma_n \to \Sigma_k$ is defined by $\overline{\mu}(A) = \mu(p^{-1}A)$, where $A \subset \Sigma_k$ is Borel. The probability measure $\overline{\mu}$ is an ergodic shift invariant probability measure on Σ_k.

The measure μ projects to a measure $\nu_\lambda = \Pi_\lambda(\mu)$ on Λ_λ. The Hausdorff dimension $\dim_H(\nu_\lambda)$ of ν_λ is defined to be the infimum of the Hausdorff dimension of Borel sets of full ν_λ-measure. The projection $\overline{\nu}_\lambda = \pi(\nu_\lambda)$ of the measure defined by $\overline{\nu}_\lambda(B) = \nu_\lambda(B \times \mathbb{R})$, where $B \subset \mathbb{R}$ is a Borel subset. We can also write $\overline{\nu}_\lambda = \overline{\Pi}_\lambda(\overline{\mu})$.

Let $h(\mu)$ denote the entropy of $\sigma : (\Sigma_n, \mu) \to (\Sigma_n, \mu)$ and let $h(\overline{\mu})$ denote the entropy of $\sigma : (\Sigma_k, \overline{\mu}) \to (\Sigma_k, \overline{\mu})$ [22]. Our main technical result is the following.

Theorem 3. *For almost all* $\lambda \in \left[\frac{1}{k}, \frac{1}{\sqrt{n}}\right]$ *we have that:*

$$\begin{aligned} \dim_H(\nu_\lambda) &= -\frac{h(\mu)}{\log\lambda} \text{ if } \max\left\{-\frac{h(\overline{\mu})}{\log\lambda}, -\frac{h(\mu)-h(\overline{\mu})}{\log\lambda}\right\} \le 1; \\ \dim_H(\nu_\lambda) &\in \left[\min\left\{1-\frac{h(\overline{\mu})}{\log\lambda}, 1-\frac{h(\mu)-h(\overline{\mu})}{\log\lambda}\right\}, -\frac{h(\mu)}{\log\lambda}\right] \text{ otherwise }. \end{aligned} \tag{1.1}$$

Remark. Assume that there are two adjacent squares in the carpet. It is easy to show that there are a dense set of values $\mathcal{E} \subset \left[\frac{1}{k}, \frac{1}{\sqrt{n}}\right]$ such that for $\lambda \in \mathcal{E}$ we have $\dim_H(\Lambda_\lambda) < -\frac{\log n}{\log\lambda}$. More precisely, for N suitably large, appropriately small changes in λ can cause two Nth level squares (of size λ^N) to coincide. This results in a drop in the dimension. This suffices to show that $\dim_H(\nu_\lambda) < -\frac{h(\mu)}{\log\lambda}$ for any fully supported measure cf. [19].

2. Hausdorff dimension, projections and transversality

In this section we recall some definitions and basic properties. Given $\delta, \epsilon > 0$ we can define

$$H^\delta_\epsilon(\Lambda) = \inf_{\{U_i\}}\left\{\sum_i (\mathrm{diam}(U_i))^\delta\right\},$$

where the infimum is over all covers $\{U_i\}$ for Λ where $\sup_i\{\mathrm{diam}(U_i)\} \le \epsilon$. The δ-dimensional Hausdorff dimension of Λ is defined by $H^\delta(\Lambda) = \lim_{\epsilon\searrow 0} H^\delta_\epsilon(\Lambda)$. Finally, the Hausdorff dimension of Λ is defined by

$$\dim_H(\Lambda) = \inf\left\{H^\delta(\Lambda) = 0\right\}.$$

A key technical device is transversality. This was first introduced in [16], but subsequently refined and developed by Peres, Solomyak and others [20], [13]. The following version is useful in the sequel.

Proposition 2.1 [13]. *Given* $k \ge 2$ *and* $0 < s < 1$ *there exists* $b_k > \frac{1}{k}$ *and* $K = K(s) > 0$ *such that for*

(i) *any sequence* $a_n \in \{-k, \dots, k\}$, $n \ge 1$; *and*
(ii) *any* $a_0 \in \{-k, \dots, k\} - \{0\}$,

we have that

$$\left|\int_0^{b_k} \frac{d\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^s}\right| \le K.$$

The first few values of b_n can be estimated numerically [13]: $b_1 = 0.649\dots$, $b_2 = 0.5$, $b_3 = 0.427\dots$, $b_4 = 0.371\dots$, $b_5 = 0.325\dots$ and afterwards $b_n = (1+\sqrt{n})^{-1}$.

The dimension of the one-dimensional measure $\overline{\nu}$ has been studied by Simon, Solomyak and Urbanski, who showed the following result.

Proposition 2.2 [20]. *For almost all* $0 < \lambda < b_{k-1}$ *we have that*

$$dim_H(\overline{\nu}) = \min\left\{1, -\frac{h(\overline{\mu})}{\log\lambda}\right\}.$$

3. Conditional entropy

We begin by recalling a few basic properties of the entropy of an invariant measure μ. Let $\alpha = \{[0], [1], \dots, [n]\}$ be the standard generating partition for Σ_n. Given $N \geq 1$, we can associate $x \in \Sigma_n$ to a cylinder

$$[x_0, \dots, x_{N-1}] = \{y \in \Sigma_n : y_j = x_j, 0 \leq j \leq N-1\}$$

of length N. We denote by $\vee_{i=0}^{N-1}\alpha$ the partition consisting of all such cylinders and entropy $H_\mu\left(\vee_{i=0}^{N-1}\alpha\right)$. The entropy $h(\mu)$ of $\sigma : (\Sigma_n, \mu) \to (\Sigma_n, \mu)$ is defined by

$$h(\mu) = \lim_{n \to +\infty} \frac{1}{N} H_\mu\left(\vee_{i=0}^{N-1}\alpha\right)$$

The asymptotic measure of a cylinder is given by the Shannon-McMillan-Brieman Theorem, i.e., For a.e. (μ) $x \in \Sigma_n$,

$$h(\mu) = -\lim_{N \to +\infty} \frac{1}{N} \log \mu\left([x_0, \dots, x_{N-1}]\right)$$

[22], [14, p.261].

Let $\mathcal{B}(\Sigma_n)$ be the Borel sigma algebra for Σ_n, and let $\mathcal{B}(\Sigma_k)$ be the Borel sigma algebra for Σ_k. Let $\mathcal{A} = p^{-1}\mathcal{B}(\Sigma_k) \subset \mathcal{B}(\Sigma_n)$ be the corresponding σ-invariant sub-sigma algebra (i.e., the sigma algebra which does not distinguish between the symbols in $\{1, \dots, n\}$ which project to the same symbol in $\{1, \dots, k\}$).

We let $H_\mu(P|\mathcal{C})$ denote conditional entropy of a partition P, with respect to a sigma algebra $\mathcal{C}$.

Notation. The conditional entropy of $\sigma : (\Sigma_n, \mu) \to (\Sigma_n, \mu)$ with respect $\mathcal{A}$ is given by

$$h(\mu|\mathcal{A}) = \lim_{N \to +\infty} \frac{1}{N} H_\mu\left(\vee_{i=0}^{N-1}\alpha|\mathcal{A}\right).$$

[1],[15], [10]. In particular, $h(\mu|\mathcal{A}) \leq h(\mu)$.

We can unique decompose the probability measure μ by

$$\mu(A) = \int \mu_\xi(p^{-1}\xi \cap A) d\overline{\mu}(\xi)$$

for any Borel set $A \subset \Sigma_n$ [18, §1.7], where we denote by μ_ξ the conditional probability measures on the fibres $p^{-1}(\xi)$ $(\xi \in \Sigma_k)$. A set $X \subset \Sigma_n$ satisfies $\mu(X) = 1$ precisely when there is a set $Y \subset \Sigma_k$ with $\overline{\mu}(Y) = 1$ such that $\mu_\xi(p^{-1}\xi \cap X) = 1$, for all $\xi \in Y$. The following result can be viewed as an analogue of the Shannon-McMillan-Brieman Theorem on the fibres $p^{-1}(\xi)$, and is appears in the work of Ledrappier and Young [9].

Proposition 3.1 (cf. [9]). *For a.e.(μ) $x \in \Sigma_n$*

$$\lim_{N \to 0} -\frac{\log \mu_\xi([x_0, \dots, x_{N-1}] \cap p^{-1}(\xi))}{N} = h(\mu|\mathcal{A}).$$

(Equivalently, the result holds for a.e.$(\overline{\mu})$ $\xi \in \Sigma_k$ and a.e.(μ_ξ) $x \in p^{-1}(\xi)$.)

Proof. We briefly recall the idea of the proof (cf. [9, Lemma 9.3.1]). We can identify $\vee_{i=0}^{N-1}\sigma^{-i}\alpha$ with the partition of cylinders of length N and

$$-\log \mu_\xi([x_0,\dots,x_N]\cap p^{-1}\xi) = I(\vee_{i=0}^{N-1}\sigma^{-i}\alpha|\mathcal{A})(x),$$

where $I(\cdot|\cdot)$ denotes the usual conditional information function. We see that

$$\begin{aligned}\frac{1}{N}I(\vee_{i=0}^{N-1}\sigma^{-i}\alpha|\mathcal{A}) &= \frac{1}{n}\sum_{k=0}^{N-1} I\left(\sigma^{-k}\alpha|\mathcal{A}\vee\left(\vee_{i=k}^{N-1}\sigma^{-i}\alpha\right)\right)\\ &= \frac{1}{n}\sum_{k=0}^{N-1} I\left(\alpha|T^k\mathcal{A}\vee\left(\vee_{i=1}^{N-k-1}\sigma^{-i}\alpha\right)\right)\circ\sigma^k\\ &\to \int I\left(\alpha|\mathcal{A}\vee\left(\vee_{i=1}^{\infty}\sigma^{-i}\alpha\right)\right)d\mu,\ \text{a.e.}(\mu)\end{aligned}$$

by the Martingale Theorem [14, p.262] and using $T\mathcal{A}=\mathcal{A}$. Finally, we observe that the limit can be identified with $h(\mu|\mathcal{A}) = H\left(\alpha|\mathcal{A}\vee\left(\vee_{i=1}^{\infty}\sigma^{-i}\alpha\right)\right)$, as required. □

Let $\epsilon,\delta,\eta>0$. By Proposition 3.1 we can choose a set $X\subset\Sigma_n$ with $\mu(X)>1-\delta$ and $K>0$ such that for $x\in X$ we have that:

$$\mu_\xi[x_0,\dots,x_N]\le K\exp\left(-\left(h(\mu|\mathcal{A})-\epsilon\right)N\right),\ \text{for } N\ge 1.$$

We can denote $X_\eta=\{\xi\in\Sigma_k : \mu_\xi(p^{-1}\xi\cap X)\ge 1-\eta\}$, then $\eta(1-\overline{\mu}(Y_\eta))<\delta$, i.e., $1-\frac{\delta}{\eta}<\overline{\mu}(X_\eta)$.

Finally, we recall a classical result that relates the entropies of μ and $\overline{\mu}$.

Proposition 3.2 (Abramov-Rohlin). $h(\mu)=h(\overline{\mu})+h(\mu|\mathcal{A})$.

(cf. [1], [14, p.256], [10], [3]).

4. Dimension of the induced measure on fibres

Let $L_y=\{(x,y): x\in\mathbb{R}\}$ denote the horizontal line at height y. Given $\xi\in\Sigma_k$, we can use the conditional measure μ_ξ on $p^{-1}(\xi)$ to define a measure $\nu_{\lambda,\xi}$ on the line $L_{\overline{\Pi}_\lambda(\xi)}$ by $\nu_{\lambda,\xi}=\Pi_\lambda(\mu_\xi)$. Our main result in this section is the following.

Proposition 4.1. *For almost every $\lambda\in[\frac{1}{k},b_{k-1}]$ there exists a set $Y\subset\mathbb{R}$ with $dim_H(Y)=dim_H(\nu_\lambda)$ such that for any $\xi\in(\overline{\Pi}_\lambda)^{-1}Y\subset\Sigma_k$ we can bound*

$$dim_H(\nu_{\lambda,\xi})\ge\min\left\{-\frac{h(\nu|\mathcal{A})}{\log\lambda},1\right\}.$$

Proof. It suffices to show that, given $\delta>0$, for almost all $\lambda\in[\frac{1}{k},b_1]$ there exists a set $X=X_\delta\subset\Sigma_k$ with $\overline{\mu}(X)\ge 1-\delta$ such that for any $\xi\in X$, $\dim(\nu_{\xi,\lambda})\ge\frac{-h(\mu|\mathcal{A})}{\log\lambda}$. In particular, we can take $Y=\cap_{n=1}^{\infty}X_{\frac{1}{n}}$.

Fix $\epsilon,\epsilon'>0$. There exists a set $X_{\epsilon'}\subset\Sigma_k$ and a constant $K>0$ such that $\overline{\mu}(X_{\epsilon'})>1-\epsilon'$ and for any $\underline{\xi}\in X_{\epsilon'}$ there exists $Y_{\epsilon'}$ such that for any $\underline{x}\in X_{\epsilon'}$:

$$\mu_\xi[x_0,\dots,x_N]\le K\exp\left(-\left(h(\mu|\mathcal{A})-\epsilon\right)N\right),\ \text{for } N\ge 1. \tag{4.1}$$

Denote $s = -\frac{h(\mu|\mathcal{A})}{\log\lambda} - 2\epsilon$. We want to apply the mass distribution method with the measure $\overline{\mu}$ restricted to $X_{\epsilon'}$ and the measure $\nu_{\lambda,\xi}$ restricted to $\Pi_\lambda(Y_{\epsilon'})$, where $\xi \in X_{\epsilon'}$. This allows us to use the explicit bound (4.1). Consider the multiple integral

$$I = \int_{\frac{1}{k}}^{b_{k-1}} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{d\nu_{\xi,\lambda}(x) d\nu_{\xi,\lambda}(y)}{|x-y|^s} \right) d\overline{\mu}(\xi) d\lambda \tag{4.2}$$

We want to prove finiteness by lifting $\nu_{\xi,\lambda}$ to μ_ξ on $p^{-1}\xi$ and then using Fubini's Theorem to rewrite the integral as:

$$\begin{aligned} I &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{b_{k-1}} \frac{d\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \\ &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{b_{k-1}} \frac{d\lambda}{|\sum_{n=1}^{\infty}(i_n - j_n)\lambda^n|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \\ &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{X_{\epsilon'}} \int_{\frac{1}{k}}^{b_{k-1}} \frac{d\lambda}{e^{(h(\mu|\mathcal{A})-2\epsilon)|\underline{i}\wedge\underline{j}|} |\sum_{n=0}^{\infty} a_n\lambda^n|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \end{aligned}$$

where we denote

$$|\underline{i} \wedge \underline{j}| = \min\{l : i_s = j_s, 0 \le s \le l\},$$

and we have that $a_n \in \{0, \pm 1, \ldots, \pm(k-1)\}$ and $a_0 \neq 0$. Thus we can use transversality (Proposition 2.1) to write

$$\begin{aligned} I &\le C \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} e^{-(h(\mu|\mathcal{A})+2\epsilon)\underline{i}\wedge\underline{j}} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \\ &\le C \sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)} (\mu_\xi \times \mu_\xi) \left(\{(\underline{i}, \underline{j}) \in Y_{\epsilon'} \times Y_{\epsilon'} : i_a = j_b, 0 \le a \le m\} \right) \\ &\le CK \sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)} e^{(h(\mu|\mathcal{A})+\epsilon)m} < +\infty. \end{aligned}$$

In particular, from the finiteness of (4.2) we deduce that that for almost every $\lambda \in [\frac{1}{k}, b_{k-1}]$, there is a set $Y = Y(\lambda) \subset \Pi_\lambda(X)$ of $\overline{\nu}$ measure $1 - \epsilon'$ such that for $y \in Y$ one can choose $\xi \in \overline{\Pi}_\lambda^{-1}(y)$ such that

$$\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{d\nu_{\lambda,\xi}(x) d\nu_{\xi,\lambda}(y)}{|x-y|^s} < +\infty.$$

The mass distribution principle shows that $\dim_H(\nu_{\lambda,\xi}) \ge s$. Finally, since $\epsilon > 0$ was arbitrary, the result follows. □

The following corollary will prove particularly useful.

Corollary. $dim_H(\nu_\lambda) \ge dim_H(\overline{\nu}_\lambda) - \min\left\{\frac{h(\mu|\mathcal{A})}{\log\lambda}, 1\right\}$.

Proof. Fix $\epsilon > 0$. We can choose $X \subset \Lambda_\lambda$ with $\nu_\lambda(X) = 1$ and $\dim_H(X) < \dim_H(\nu_\lambda) + \epsilon$. Using a variant on the Marstrand Slice theorem (cf. [5], [2]) we can bound

$$\begin{aligned}\dim_H(\nu)+\epsilon &\geq \dim_H(X)\\ &\geq \dim_H(Y)-\min\left\{\frac{h(\mu|\mathcal{A})}{\log\lambda},1\right\}.\\ &\geq \dim_H(\overline{\nu}_\lambda)-\min\left\{\frac{h(\mu|\mathcal{A})}{\log\lambda},1\right\}\end{aligned}$$

Since $\epsilon>0$ is arbitrary, the result follows. □

Proof of Theorem 3. The inequality $\dim_H(\nu_\lambda)\leq -\frac{h(\mu)}{\log\lambda}$ for (2.1) is easily seen to hold for all $0<\lambda<1$. More precisely, by the Shannon-McMillan-Brieman theorem we have that

$$h(\mu)=\lim_{N\to+\infty}\frac{1}{N}\log\mu\left([x_0,\dots,x_{N-1}]\right)$$

for a.e. (μ) $x\in\Sigma_n$, where $[x_0,\dots,x_{N-1}]=\{y\in\Sigma_n : y_j=x_j, 0\leq j\leq N-1\}$. In particular,

$$\limsup_{N\to+\infty}\frac{\log\nu_\lambda(B(x,\lambda^N))}{\log\lambda^N}\leq-\frac{h(\mu)}{\log\lambda},$$

gives a bound on the pointwise dimension, and thus for the Hausdorff dimension.

To get the reverse inequality, one can compare Proposition 2.2, Proposition 3.2 and the above corollary. This completes the proof of Theorem 3. □

Proof of Theorem 1. In particular, if we let $\mu=\left(\frac{1}{n},\cdots,\frac{1}{n}\right)^{\mathbb{Z}^+}$ be the standard Bernoulli measure then $h(\mu)=\log n$. In particular, Theorem 1 follows from Theorem 3. □

5. Sets of positive measure, projections and transversality

We recall that a measure ν_λ on $\mathbb{R}^d$ is absolutely continuous if for any Borel set $A\subset\mathbb{R}^d$ satisfying $\mathrm{leb}(A)=0$ necessarily also satisfies $\nu_\lambda(A)=0$. The absolute continuity of the measure $\overline{\nu}_\lambda$ (where $d=1$) has been studied by Simon, Solomyak and Urbanski [20], who showed the following nice result.

Proposition 5.1 (Simon, Solomyak and Urbanski). *For almost all λ in the set*

$$\left\{\left[\frac{1}{k},b_{k-1}\right] : h(\overline{\mu})\geq-\log\lambda\right\}$$

the measure $\overline{v}_\lambda=\overline{\Pi}_\lambda\overline{\mu}$ is absolutely continuous.

A key ingredient in the proof of 5.1 is the following application of the transversality technique.

Lemma 5.2, [13]. *Let $\xi\in\Sigma_k$. There exists $C>0$ such that if $\underline{i},\underline{j}\in p^{-1}(\xi)$ then for $a>0$*

$$\mathrm{leb}\left\{\lambda\in(a,b_{k-1}) : |\overline{\Pi}_\lambda(\underline{i})-\overline{\Pi}_\lambda(\underline{j})|\leq\epsilon\right\}\leq C\left(a^{-|\underline{i}\wedge\underline{j}|}\right)\epsilon.$$

The following result should be viewed as a two dimensional version of Proposition 5.1.

Theorem 4. *For almost all λ in the set*

$$\left\{\left[\frac{1}{k}, b_{k-1}\right] \ : \ \min\{h(\mu|\mathcal{A}), h(\overline{\mu})\} \geq -\log\lambda\right\}$$

the measure $v_\lambda = \Pi_\lambda\mu$ is absolutely continuous.

For such λ the set Λ_λ has positive Lebesgue measure for such λ. In particular, Theorem 2 then follows from Theorem 4.

A key ingredient ingredient in the proof of Theorem 4 is show that typical measures ν_ξ are absolutely continuous on $L_{\overline{\Pi}_\lambda(\xi)}$. This is contained in the following result.

Proposition 5.3. *For almost all λ in*

$$\left\{\lambda \in \left[\frac{1}{k}, b_{k-1}\right] \ : \ h(\mu|\mathcal{A}) > -\log\lambda\right\}$$

there exists a set $X \subseteq \Sigma_k$ such that $\overline{\mu}(X) = 1$ and for any $\xi \in X$ the measure $\nu_{\lambda,\epsilon}$ is absolutely continuous on $L_{\overline{\Pi}_\lambda(\xi)}$.

Proof. It suffices to show that given $\epsilon' > 0$, there exists a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'}) \geq 1 - \epsilon'$ and for any $\xi \in X_{\epsilon'}$ there exists a set $Y_{\epsilon',\xi} \subset L_{\overline{\Pi}_\lambda(\xi)}$ where $\mu_\xi(Y'_\epsilon) \geq 1 - \epsilon'$ and $\nu_{\lambda,\epsilon}$ is absolutely continuous on $Y_{\epsilon',\xi}$. We can then take $X = \cap_{n=1}^\infty X_{\frac{1}{n}}$.

Let $\epsilon, \epsilon' > 0$. From Proposition 3.1 we know that there exists $K > 0$ and a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'})$ and for $\xi \in X_{\epsilon'}$ there exists $Y_{\epsilon',\xi} \subseteq p^{-1}\xi$ with $\mu_\xi(Y_{\epsilon',\xi}) > 1 - \epsilon'$ and for $\underline{x} \in Y_{\epsilon',\xi}$ equation (3.1) holds, i.e.,

$$\mu_\xi[x_0, \ldots, x_{N-1}] \leq K\exp\left(-\left(h(\mu|\mathcal{A}) - \epsilon\right)N\right), \text{ for } N \geq 1 \qquad (5.1).$$

We recall that the lower pointwise density for $\nu_{\lambda,\xi}$ (restricted to $\Pi_\lambda Y_{\epsilon',\xi}$) is defined by

$$\underline{D}(\nu_\xi)(x) = \liminf_{\epsilon\searrow 0} \frac{\nu_\xi(B(x,\epsilon) \cap \Pi_\lambda Y_{\epsilon',\xi})}{2\epsilon}.$$

To show that ν_ξ is absolutely continuous it suffices to show that $\underline{D}(\nu_\xi)(x)$ is finite, for a.e.$(\nu_{\xi,\lambda})$ $x \in \Pi_\lambda Y_{\epsilon',\xi}$. In particular, it suffices to show that

$$\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) d\nu_{\xi,\lambda}(x) < +\infty.$$

Moreover, to show that for almost every λ there exists a set of ξ of $\overline{\mu}$ measure at least $1 - \epsilon'$ such that $\nu_{\xi,\lambda}$ is absolutely continuous, it suffices to show that

$$I := \int_t^{b_{k-1}} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) d\nu_{\xi,\lambda}(x)\right) d\overline{\mu}(\xi) d\lambda < +\infty,$$

providing t is sufficiently large. We take $t > e^{h(\mu|\mathcal{A})+2\epsilon}$. For $\omega, \tau \in p^{-1}\xi$ we define

$$\phi_r(\omega, \tau) = \{\lambda : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\},$$

for $r > 0$. We start by lifting to the shift space, applying Fatou's Lemma and Fubini's Theorem

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r}\int_t^{b_{k-1}}\int_{X_{\epsilon'}}\int_{Y_{\epsilon',\xi}}\int_{Y_{\epsilon',\xi}} \chi(\omega,\tau)\omega,\tau\mu_\xi(\omega)d\mu_\xi(\tau)d\overline{\mu}(\xi)d\lambda \\ &\leq \liminf_{r\to 0}\frac{1}{2r}\int_{X_{\epsilon'}}\int_{Y_{\epsilon',\xi}}\int_{Y_{\epsilon',\xi}} \mathrm{leb}(\phi_r(\omega,\tau))d\mu_\xi(\omega)d\mu_\xi(\tau)d\overline{\mu}(\xi), \end{aligned}$$

where χ is the characteristic function for $\{(\omega,\tau) : |\Pi_\lambda(\omega)-\Pi_\lambda(\tau)|\leq r\}$. We can apply Lemma 5.2 and then use Equation (5.1).

$$\begin{aligned} I &\leq C\int_{X_{\epsilon'}}\int_{Y_{\epsilon',\xi}}\int_{Y_{\epsilon',\xi}} t^{-|\omega\wedge\tau|}d\mu_\xi(\omega)d\mu_\xi(\tau)d\overline{\mu}(\xi) \\ &\leq C\int_{X_{\epsilon'}}\int_{Y_{\epsilon',\xi}}\int_{Y_{\epsilon',\xi}} e^{-|\omega\wedge\tau|(h(\mu|\mathcal{A})+2\epsilon)}d\mu_\xi(\omega)d\mu_\xi(\tau)d\overline{\mu}(\xi) \\ &\leq C\int_{X_{\epsilon'}}\sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)}(\mu_\xi\times\mu_\xi)(\Delta_m)\,d\overline{\mu}(\xi) \\ &\leq CK\sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)}e^{m(h(\mu|\mathcal{A})+\epsilon)} < \infty, \end{aligned}$$

where $\Delta_m = \{(\tau,\omega)\in Y_{\epsilon',\xi}\times Y_{\epsilon',\xi} : \omega_1=\tau_1,\dots,\omega_m=\tau_m\}$. This completes the proof. □

Proof of Theorem 4. By Propositions 5.1 and 5.2 it follows that $\overline{\nu}_\lambda$ is absolutely continuous and there exists a set, $X_\lambda \subseteq \Sigma_k$ such that for all $\xi\in X_\lambda$, $\nu_{\xi,\lambda}$ is absolutely continuous on $L_{\Pi_\lambda(\xi)}$ for almost all λ. We choose λ to satisfy these properties. Let $Y\subseteq\mathbb{R}^2$ be a set such that $\mathrm{leb}^2(Y)=0$. Let $G=\{y\in\mathbb{R} : \mathrm{leb}(L_y)=0\}$ then it is clear that $\mathrm{leb}(G)=0$. Using the decomposition of μ,

$$\nu_\lambda(Y)=\mu(\Pi_\lambda^{-1}(Y)) = \int \mu_\xi(\Pi_\lambda^{-1}(Y)\cap p^{-1}\xi)\mathrm{d}\overline{\mu}(\xi) = \int_{X_\lambda}\mu_\xi(\Pi_\lambda^{-1}(Y)\cap p^{-1}\xi)\mathrm{d}\overline{\mu}(\xi).$$

Let $\Gamma=\{\xi\in X_\lambda : \mu_\xi(\Pi_\lambda^{-1}(Y)\cap p^{-1}\xi)>0\}$. If $\xi\in\Gamma$ then $\nu_{\lambda,\xi}(Y\cap L_{\Pi_\lambda\xi})>0$ and hence $\mathrm{leb}(Y\cap L_{\Pi_\lambda\xi}>0$. Thus if $\xi\in\Gamma$ then $\overline{\Pi}_\lambda\xi\in G$. From the absolute continuity of $\overline{\nu}_\lambda$ it follows that $\overline{\nu}_\lambda(G)=0$ and so $\overline{\mu}(\Gamma)=0$. Hence $\nu_\lambda(Y)=0$ and it follows that ν_λ is absolutely continuous. □

Remark 1. In order to derive the bound in (0.3), we can consider the Bernoulli measure $\overline{\mu}=(q_1,\dots,q_k))^{\mathbb{Z}^+}$ on Σ_k. Let $\mu=(\frac{q_1}{n_1},\dots,\frac{q_k}{n_k})^{\mathbb{Z}^+}$ be the Bernoulli measure on Σ_n. We have that $h(\overline{\mu})=-\sum_{j=1}^k q_j\log q_j$ and $h(\mu)=-\sum_{j=1}^k q_j\log\left(\frac{q_j}{n_j}\right)$. In particular, we see that

$$h(\mu|\mathcal{A}) = h(\mu)-h(\overline{\mu}) = \sum_{j=1}^k q_j\log n_j.$$

For any $(q_1,\dots,q_k)$ such that $h(\overline{\mu})=h(\mu|\mathcal{A})$ we could might choose $t=e^{-h(\overline{\mu})}$ in Theorem 3.

Remark 2. If we consider μ supported on some subshift Σ on k-symbols then there is a possibility that the transversality constant can be increased. This was considered by Solomyak [21]. For example, by recoding by words of length 3 we can restrict to symbols of the form $[*, *, 0]$ to get a subshift $\widehat{\sigma} : \Sigma_A \to \Sigma_A$. This reduces the entropy to $h(\widehat{\sigma}) = \frac{2}{3}h(\sigma)$. However, the advantage is that the transversality constant b_{k-1} is also increased, to $\widehat{b}_{k-1} > \frac{2}{3}$ [21]. This technique allows us to extend the absolute continuity results to larger domains of λ

6. Comments and Generalizations

6.1 Limitations on the estimates. It is easy to construct examples for which there are examples for which one can find an open interval $U \subset [\frac{1}{k}, \frac{1}{\sqrt{n}}]$ for which $\dim_H(\Lambda_\lambda) < -\frac{\log n}{\log \lambda}$ for $\lambda \in U$.

Example 4. Let $k = 3$ and let $c_1, \ldots, c_5$ correspond to the four corners, plus $(1, 0)$ square. In this case, $s = \frac{3^{-3} * 2^{-2}}{5} = 0.3920\ldots$. However for $\lambda > 0.4$ we have that $\dim_H(\Lambda_\lambda) \le 1 - \frac{\log 2}{\log \lambda} < -\frac{\log 5}{\log \lambda}$. For $\lambda > 0.4082\ldots$ we let μ be the $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$-Bernoulli measure on Σ_5 and ν_λ the projection of μ onto Λ_λ.. Theorem 3 gives $\dim_H(\nu_\lambda) \ge 1 - \frac{\log 2}{\log \lambda}$ for a.e. $\lambda > 0.4082\ldots$ and thus $\dim_H(\Lambda_\lambda) = 1 - \frac{\log 2}{\log \lambda}$ for a.e. $\lambda > 0.4082\ldots$.

6.2 More general contractions. It is easy to see (using an affine transformation of the plane) that we can consider more general grids $\mathcal{C}$ by translating horizontally each row by the same amount. More generally, we can consider parameterised families contractions $T_{ij}^{(\lambda)} : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane given by

$$T_{ij} : (x, y) \mapsto (f_i^{(\lambda)}(x), g_{ij}^{(\lambda)}(x, y))$$

where $f_i^{(\lambda)} : [0, 1] \to [0, 1]$ and $g_{ij}^{(\lambda)} : [0, 1] \times [0, 1] \to [0, 1]$ are C^∞ contractions. An important feature is that the foliation of the plane by vertical lines $\{x\} \times \mathbb{R}$, for $x \in \mathbb{R}$, is preserved under the maps, i.e., $T_{ij}^{(\lambda)} L_x = L_{f_i^{(\lambda)}(x)}$.

Let Σ_n be the space of sequences with symbols (i, j) and let $\sigma : \Sigma_n \to \Sigma_n$ be the full shift on n symbols defined by $(\sigma x)_n = x_{n+1}$. Let $\Pi_\lambda : \Sigma_n \to \mathbb{R}^2$, where

$$\Pi_\lambda(\underline{x}) = \sum_{m=1}^{\infty} T_{x_1}^{(\lambda)} \cdots T_{x_m}^{(\lambda)}(0, 0),$$

say, is the natural map to the associated attractor Λ_λ. Let Σ_k be the space of sequences with symbols i and let $\sigma : \Sigma_k \to \Sigma_k$ be the coding corresponding to the iterated function scheme $\{f_i^{(\lambda)}\}_{i=1}^k$ and let $\overline{\Pi}_\lambda : \Sigma_k \to \overline{\Lambda}_\lambda$ be the associated map. Let μ be an ergodic probability measure on Σ_n and let $\overline{\mu}$ be the corresponding ergodic probability measure on Σ_k. We decompose μ as in Section 3, that is for any Borel set $A \subseteq \Sigma_n$, $\mu(A) = \int_{\Sigma_k} \mu_\xi(A) d\overline{\mu}(\xi)$ and denote Π_λ as the restriction of Π_λ to $p^{-1}\xi$. Thus we can define measures $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$ on Λ_λ, $\overline{\nu}_\lambda = \overline{\mu} \circ \overline{\Pi}_\lambda^{-1}$ on $\overline{\Lambda}_\lambda$ and $\nu_{\lambda,\xi} = \mu_\xi \circ \Pi_\lambda^{-1}$ on $L_{\overline{\Pi}_\lambda \xi}$, respectively. We can associate two Lyapunov exponents

$$\chi_1 = \int_{\Sigma_k} \log |f'_{x_0}| \circ \overline{\Pi}_\lambda d\overline{\mu}(x) \text{ and } \chi_2 = \int_{\Sigma_k} \int_{p^{-1}\xi} \log |\frac{\partial g_{x_0}}{\partial y}| \circ \Pi_\lambda d\mu_\xi(\underline{x}) d\overline{\mu}(\xi).$$

We need the following transversality conditions to be satisfied: There exists a constant $C_1 > 0$ so that for $\omega, \tau \in \Sigma_k$ with $\omega_0 \neq \tau_0$:

$$\text{leb}\{\lambda \in U : |\overline{\Pi}_\lambda(\omega) - \overline{\Pi}_\lambda(\tau)| \leq r\} \leq C_1 r;$$

and for any $\xi \in \Sigma_k$ there exists a constant C_2 so that for $\underline{\omega}, \underline{\tau} \in \ p^{-1}\xi$:

$$\text{leb}\{\lambda \in U : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\} \leq C_2 r.$$

We let A_λ be the set where both transversality condiitions are satisfied. We can now state analogues to Theorem 3.

Proposition 6.1. *For almost all $\lambda \in A_\lambda$:*

$$\dim_H(\nu_\lambda) \geq -\left(\frac{h(\overline{\mu})}{\chi_1} + \frac{h(\mu|\mathcal{A})}{\chi_2}\right) \text{ if } \max\left\{-\frac{h(\overline{\mu})}{\chi_1}, -\frac{-h(\mu|\mathcal{A})}{\chi_2}\right\} \leq 1; \text{ and}$$
$$\dim_H(\nu_\lambda) \geq 1 + \min\left\{-\frac{-h(\overline{\mu})}{\chi_1}, -\frac{h(\mu|\mathcal{A})}{\chi_2}\right\} \text{ otherwise.}$$

The analogue of Theorem 4 is the following.

Proposition 6.2. *For almost all λ in the set,*

$$\left\{\lambda \in A_\lambda : \min\left\{-\frac{h(\overline{\mu})}{\chi_1}, -\frac{h(\mu|\mathcal{A})}{\chi_2}\right\} \geq 1\right\}$$

ν_λ is absolutely continuous.

Application (Bedford-McMullen). This setting includes the generalized Sierpinski carpet studied by McMullen and Bedford. Let $k.m \geq 2$ and write $\beta = \frac{\log k}{\log m}$. Consider contractions

$$T_i : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T_i : (x, y) \mapsto (\lambda x, \lambda^\beta y) + c_i,$$

where $c_i \in \{(j, l) : 0 \leq j \leq k-1, 0 \leq l \leq m-1\}$, $i = 1, \dots, n$ are distinct points. Let $\Lambda_{\lambda,\lambda^\beta}$ be the associated limit set. In the particular case that $\lambda = \frac{1}{m}$ and $\lambda^\beta = \frac{1}{k}$ this corresponds to the generalized Sierpinski carpet construction of McMullen and Bedford. The same general method allows one to show there exists $s > \frac{1}{k}$ such that for almost all $\lambda \in [\frac{1}{k}, s]$ we have that

$$\dim(\Lambda_{\lambda,\lambda^\beta} = \frac{\log\left(\sum_{i=0}^{k-1} n_i^\beta\right)}{-\log \lambda}$$

where $n_i = \text{Card}\{c_i : c_i^{(1)} = i\}$. To see this is a lower bound let $p_i = \frac{n_{c_i^{(1)}}^{\beta-1}}{\sum_{j=0}^{k-1} n_i^\beta}$ for $i = 1, \dots, n$ and μ be $(p_1, \dots, p_n)$ Bernoulli measure on Σ_n. If we let ν_λ be the natural projection of μ onto $\Lambda_{\lambda,\lambda^\beta}$ then Proposition 6.1 can be used to show that there exists s such that

$$\dim \nu_\lambda \geq \frac{\log\left(\sum_{i=0}^{k-1} n_i^\beta\right)}{-\log \lambda}$$

for almost every $\lambda \in [\frac{1}{k}, s]$. A simple adjustment of Mcmullen's argument to account for the overlaps shows that this is also an upper bound.

In [14] Peres and Solomyak ask whether it is possible to find an example of a self-similar set with positive measure but empty interior (question 2.4). We cannot answer this question[2] but we can show there exist simple examples of self affine sets in $\mathbb{R}^2$ with positive measure and empty interior. Let

$$T_i : (x, y) \to \left(\frac{1}{3}x, \lambda y\right) + c_i$$

for $1 \leq i \leq 7$ and where $c_1 = (0,0), c_2 = (0,1), c_3 = (0,2), c_4 = (1,1), c_5 = (2,0), c_6 = (2,1)$ and $c_7 = (2,2)$. Let $\Lambda(\lambda)$ be the associated limit set. We can choose μ to be the Bernoulli measure defined by the probability vector $\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right)$. Let ν_λ be the natural projection of μ onto $\Lambda(\lambda)$. In the setting of Propositions 6.1 and 6.2 we have $\chi_1 = -\log 3$, $\chi_2 = \log \lambda$, $h(\bar{\mu}) = \log 3$ and $h(\mu|\mathcal{A}) = 0.7324\ldots$. Thus, $\min\left\{-\frac{h(\bar{\mu})}{\chi_1}, -\frac{h(\mu|(A))}{\chi_2}\right\} \geq 1$ for $\lambda > 0.4807\ldots$. Since for $\lambda < \frac{1}{2}$ the transversality conditions hold, we have by the method of Proposition 6.2 that for almost all $\lambda \in (0.4807\ldots, 0.5)$ the measure ν_λ is absolutely continuous, and hence $\Lambda(\lambda)$ has positive measure.

We now need to show that $\Lambda(\lambda)$ has empty interior. Note that $\Sigma_k = \{0,1,2\}^{\mathbb{N}}$ and $\overline{\Pi}_\lambda(\underline{x}) = \sum_{n=0}^{\infty} x_n \left(\frac{1}{3}\right)^n$. Consider the set

$$A = \{x \in \mathbb{R} : \exists \underline{x} \in \{0,1,2\}^{\mathbb{N}}, N \in \mathbb{N} \text{ such that } \overline{\Pi}_\lambda(\underline{x}) = x \text{ and } \forall n \geq N, x_n = 1\}.$$

This set is clearly dense in $\overline{\Lambda(\lambda)} = [0,3]$ and for any $x \in A$ the sequence $\underline{x} \in \{0,1,2\}^{\mathbb{N}}$ such that $\overline{\Pi}_\lambda(\underline{x}) = x$ is unique. Given $x \in A$, let L_x be the vertical line intersecting $(x, 0)$. If $\underline{y} \in \Sigma_n$ and $\Pi_\lambda(\underline{y}) \in L_x$ then whenever $x_n = 1$ then necessarily $y_n = (1,1)$. However, by hypothesis $x_n = 1$ for all $n \geq N$ and thus there are only a finite number of sequences $\underline{y}$ such that $\Pi_\lambda(\underline{y}) \in L_x$. Hence $L_x \cap \Lambda(\lambda)$ contains a finite number of points for any given $x \in A$. Since A is dense in $\overline{\Lambda(\lambda)} = [0,3]$ the set $\Lambda(\lambda)$ cannot contain open sets and so has empty interior.

Remark. Most of the of the elements of the above proofs depend on entropy and are essentially measure theoretic in flavour. Thus, it is possible to extend many of these arguments to non-uniformly hyperbolic systems (e.g., parabolic points, systems which contract in mean).

6.3 Higher dimensions.. There are natural extensions to higher dimensions. Perhaps this is best illustrated by simple examples in $\mathbb{R}^3$.

Example 1. Consider the Menger sponge, consisting of 20 contractions. We can associate to the corresponding subshift Σ_{20} the Bernoulli measure μ with equal weights $\frac{1}{20}$. The Sponge projects to the Sierpinski gasket, and the measure μ projects to a Bernoulli measure $\overline{\mu}$ on Σ_8 given by

$$\overline{\mu} = \left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}\right)^{\mathbb{N}}.$$

[2] Added in proof: This question has now been answered in the appendix to this paper.

Finally, this projects to a Bernoulli measure $\widehat{\mu}$ on $\mathbb{R}$ given by $\widehat{\mu} = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)^{\mathbb{N}}$. The entropies are $h(\mu) = \log(20) = 2.9957$, $h(\overline{\mu}) = \log(5) - \frac{1}{2}\log(2) - \frac{1}{2}\log(3) = 2.099\ldots$ and $h(\widehat{\mu}) = \log(5) - \frac{4}{5}\log(2) = 1.054\ldots$. The method we described before applies providing

$$\lambda \leq \min\left\{e^{-(h(\mu)-h(\overline{\mu}))}, e^{-(h(\overline{\mu})-h(\widehat{\mu}))}, e^{-h(\widehat{\mu})}\right\} = 0.348\ldots.$$

Consider the probability vector (q_1, q_2, q_3) for $\overline{\mu}$ as in Example 2 of the introduction. If we choose the probability vector for the 20 subsquares with weights $q_1/8$ and $q_2/4$ then we see that the measure is absolutely provided $\lambda \geq 0.393\ldots$

Example 2. Consider Sierpinski tetrahedron, consisting of 4 contractions. We can associate to the corresponding subshift Σ_4 the Bernoulli measure $\mu = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^{\mathbb{N}}$. The Sponge projects to the Sierpinski gasket, and the measure μ projects to a Bernoulli measure $\overline{\mu}$ on Σ_8 given by $\overline{\mu} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. Finally, this projects to a Bernoulli measure $\widehat{\mu}$ on $\mathbb{R}$ given by $\widehat{\mu} = \left(\frac{1}{4}, \frac{3}{4}\right)^{\mathbb{N}}$. The entropies are $h(\mu) = \log(4)$, $h(\overline{\mu}) = \log(2)$ and $h(\widehat{\mu}) = \log(4) - \log(3)$. The method we described before applies providing

$$\lambda \leq \min\left\{e^{-(h(\mu)-h(\overline{\mu}))}, e^{-(h(\overline{\mu})-h(\widehat{\mu}))}, e^{-h(\widehat{\mu})}\right\} = 0.569\ldots$$

References

1. Abramov and Rohlin, *The entropy of a skew product of measure preserving transformations*, Transl. Amer. Math. Soc. (2) **48** (1966), 255-265.
2. C. Bishop, *Topics in real analysis (unpublished lecture notes)*, http://www.math.sunysb.edu/∼bishop/classes/math639.S01/math639.html.
3. T. Bogenschütz and H. Crauel, *The Abramov-Rokhlin Formula*, Springer L.N.M. 1514, 1992, pp. 32-35.
4. D. Broomhead, J. Montaldi and N. Sidorov, *Golden Gaskets: Variations on the Sierpinski sieve*, Preprint: http://www.ma.umist.edu/∼ nikita/gold-final.pdf, ???.
5. K. Falconer, *Fractal Geometry*, Wiley, London, 1990.
6. L. Jonker and J. Veerman, *Semi-continuity of dimension and measure of locally scaled fractals*, Fund. Math. **173** (2002), 113-131.
7. T. Jordan, *The dimension of fat Sierpinski triangles*, preprint.
8. F. Ledrappier, *On the dimension of some graphs*, Symbolic Dynamics and its Applications (P. Walters, ed.), Contemporary Mathematics, vol. 135, Amer. Math. Soc., Providence, 1992, pp. 285-293.
9. F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms II relations between entropy, exponents and dimension.*, Annals of Math. **112** (1985), 540-574.
10. F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations*, J. London. Math. Soc. **16** (1977), 568-576.
11. C. McMullen, *The Hausdorff dimension of general Sierpinski carpets*, Nagoya Math. J. **96** (1984), 1-9.
12. W. Parry, *Entropy and generators in ergodic theory*, W.A. Benejamin, New York, 1969.
13. Y. Peres and B. Solomyak, *Self-similar measures and the intersection of Cantor sets*, Trans. Amer. Math. Soc. **350** (1998), 4065-4087.
14. Y. Peres and B. Solomyak, *Problems on self-similar and self-affine sets; an update*, Progress in probability **46** (2000), 95-106.
15. K. Petersen, *Ergdic Theory*, C.U.P., Cambridge, 1983.
16. M. Pollicott and K. Simon, *The Hausdorff dimension of λ-expansions with deleted digits*, Trans. Amer. Math. Soc. **347** (1995), 967-983.

17. V. Rohlin, *Lectures on the entropy theory of measure preserving transformations*, Russ. Math. Soc. **22** (1967), 1-52.
18. V. Rohlin, *On the fundamental ideas of measure theory*, Transl. Amer. Math. Soc. **71** (1952), 1-54.
19. K. Simon and B. Solomyak, *On the dimension of self-similar sets*, Fractals **10**, 59-65.
20. K. Simon, B. Solomyak and M. Urbanski, *Invariant measures for parabolic IFS with overlaps and random continued fractions*, Trans. Amer. Math. Soc. **353** (2001), 5145-5164.
21. B. Solomyak, *On the random series* $\sum \pm\lambda^n$ *(an Erdös problem)*, Annals of Math. **142** (1995), 611-625.
22. P. Walters, *Ergdic Theory*, Springer, Berlin, 1982.