Mathematical models of volcanic plumes

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13 October 2011

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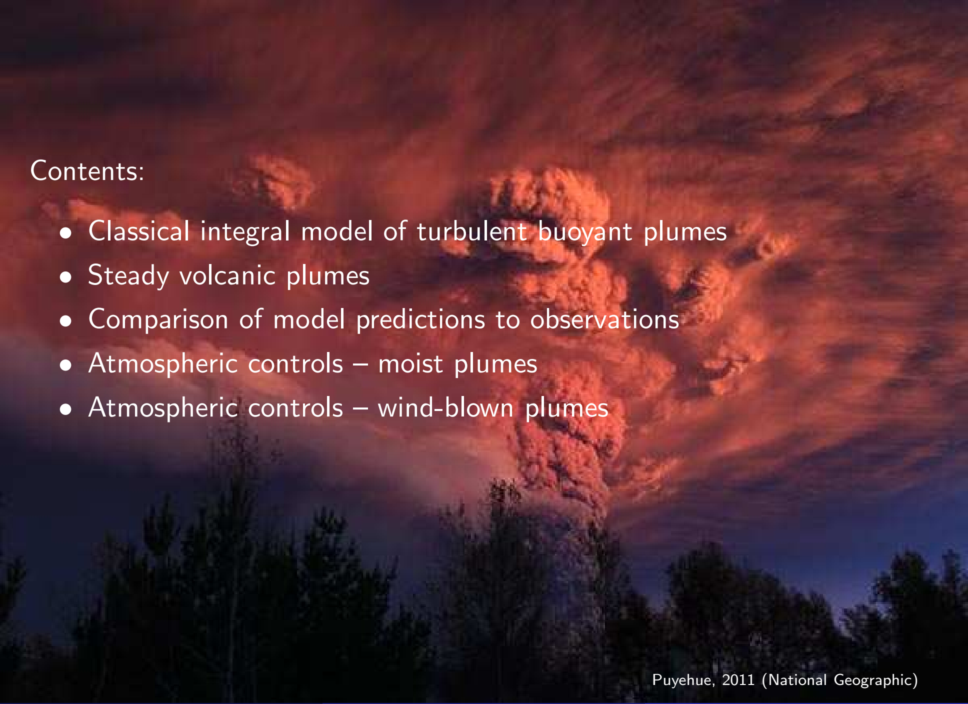
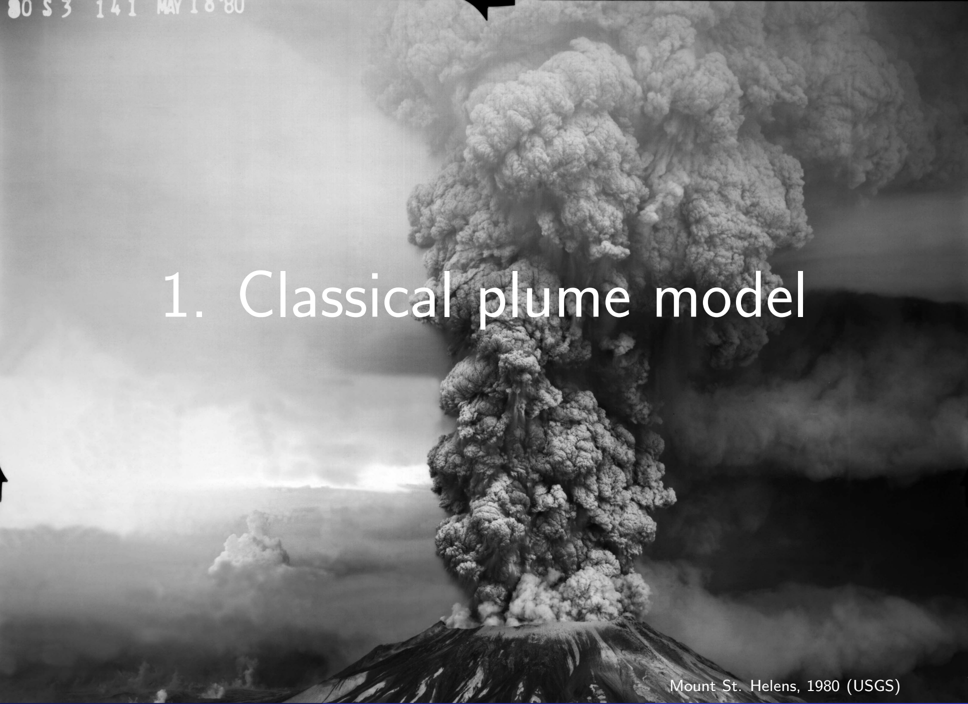


Puyehue, 2011 (National Geographic)

1. Classical plume model



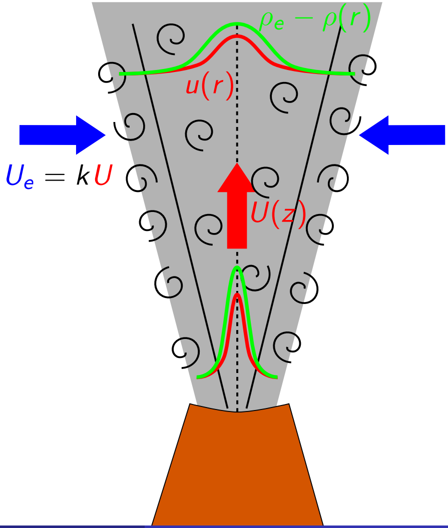
Mount St. Helens, 1980 (USGS)

Turbulent buoyant plumes

Morton, Taylor, Turner (1956) model of buoyant convection from maintained source.

Key assumptions:

- plume is steady on timescale longer than eddy turnover time;
- self-similarity of mean profiles;
- entrainment velocity proportional to a characteristic velocity of the plume;
- linear dependence of density to concentration;
- density difference is small compared to reference density (Boussinesq approximation).



Turbulent buoyant plumes

Replacing velocity and buoyancy profiles with equivalent top-hat profiles and defining:

mass flux

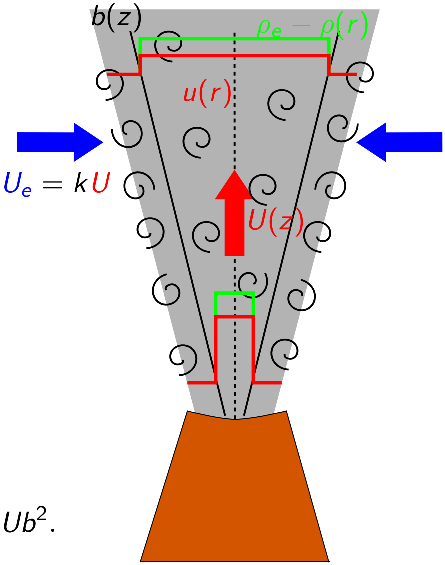
$$\pi Q = 2\pi \int \rho u r \mathrm{d}r = \pi \rho U b^2,$$

momentum flux

$$\pi M = 2\pi \int \rho u^2 r \mathrm{d}r = \pi \rho U^2 b^2,$$

buoyancy flux

$$\pi F = 2\pi \int g(\rho_e - \rho) u r \mathrm{d}r = \pi g(\rho_e - \rho) U b^2.$$



Turbulent buoyant plumes

Conservation of mass, momentum and buoyancy through a control volume leads to a system of 3 governing equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}(b^2U) = 2kUb, \qquad \frac{\mathrm{d}Q}{\mathrm{d}z} = 2k\sqrt{\rho_0 M},$$

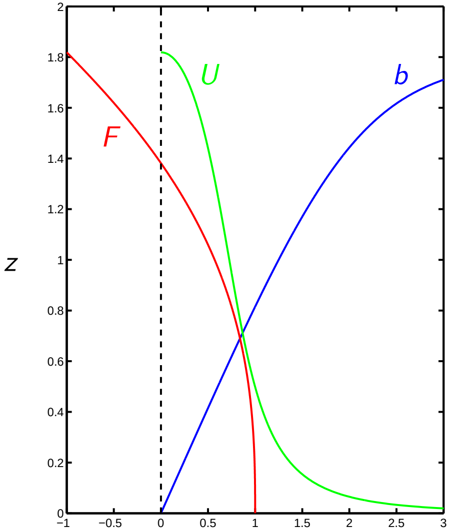
$$\frac{\mathrm{d}}{\mathrm{d}z}(b^2U^2) = b^2 g\frac{\rho_e - \rho}{\rho_0}, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{FQ}{M},$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(b^2 U g\frac{\rho_e - \rho}{\rho_0}\right) = b^2 U\frac{g}{\rho_0}\frac{\mathrm{d}\rho_e}{\mathrm{d}z}, \qquad \frac{\mathrm{d}F}{\mathrm{d}z} = -N^2 Q,$$

with ambient stratification represented by buoyancy frequency N with

$$N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_e}{\mathrm{d}z}.$$

Uniform stratification $N^2 = \mathrm{const.}$



Uniform stratification $N^2 = \mathrm{const.}$

For a pure plume there are 3 dimensional parameters: the source buoyancy flux F_0, the buoyancy frequency N and a density scale ρ_0. Therefore the rise height, H, scales as

$$H \sim (F_0/\rho_0)^{1/4} N^{-3/4}.$$

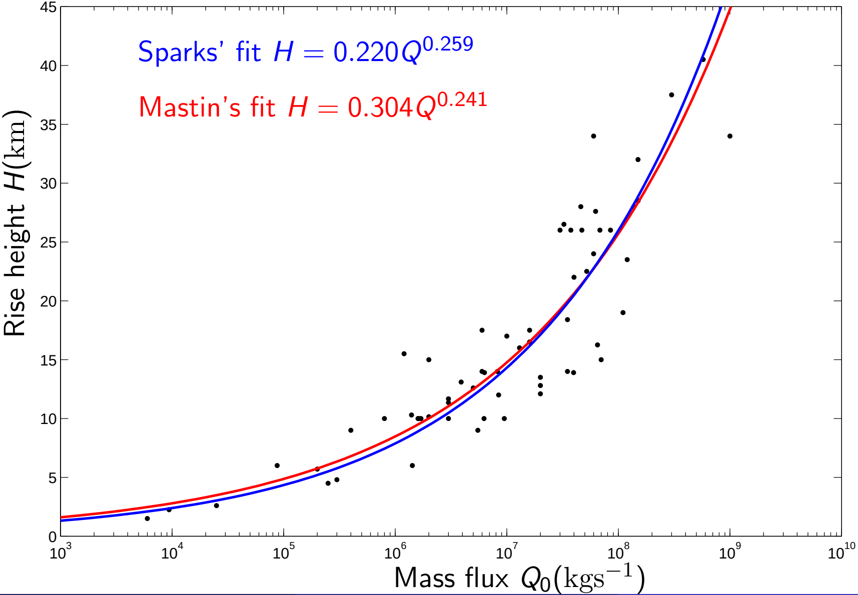
An estimate for rise height of volcanic plumes can be made by relating F_0 to source mass flux, Q_0,

$$\frac{F_0}{\rho_0} = \frac{g}{\rho_0}\left(\frac{\rho_e - \rho}{\rho_0}\right) Q_0 \approx \frac{g}{\rho_0}\left(\frac{T - T_a}{T_a}\right) Q_0,$$

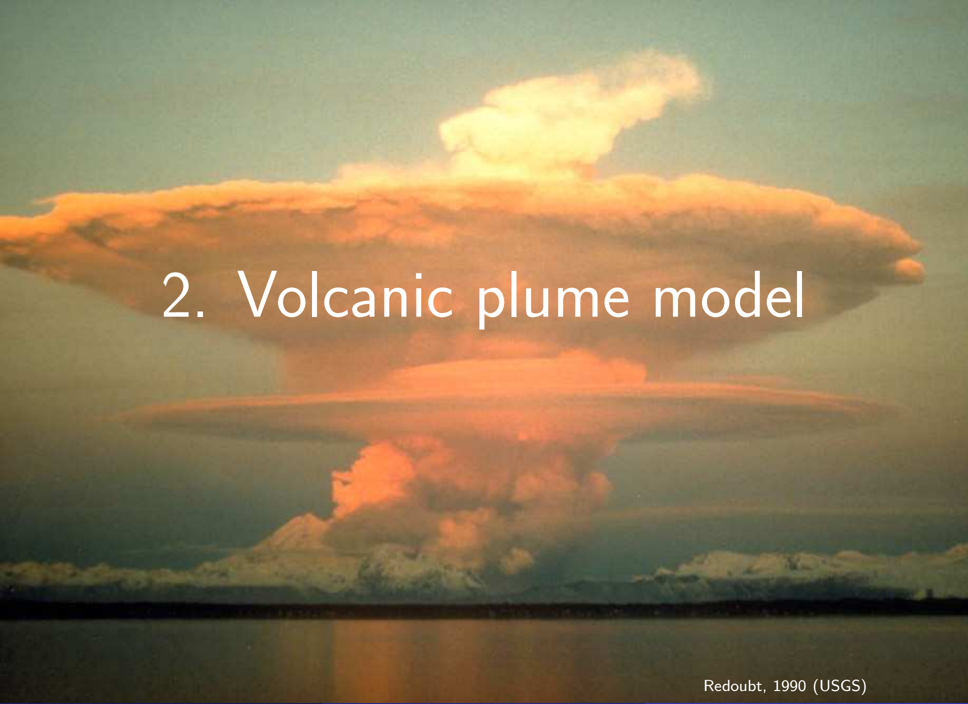
so

$$H \sim Q_0^{1/4} N^{-3/4}.$$

Comparison to volcanic plume observations

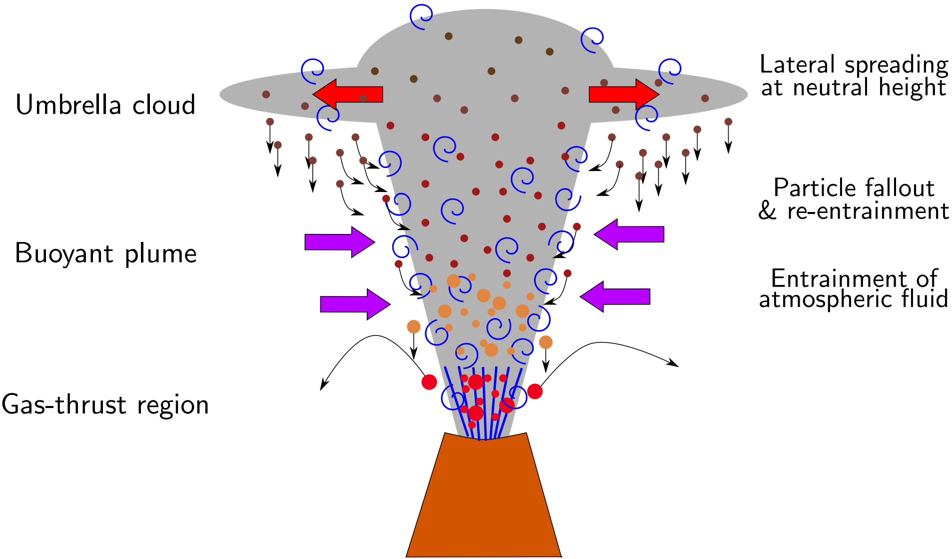


2. Volcanic plume model



Redoubt, 1990 (USGS)

Volcanic plume structure



Volcanic plume model

Woods (1988) developed a model of volcanic plumes, extending the classical Morton, Taylor, Turner model with descriptions of:

- the heat content of the erupting material
- heat enchange between particles and entrained air
- the thermodynamic expansion of the gases
- nonlinear variations of density with temperature
- varying atmospheric stratification
- different entrainment rates in jet and plume regions

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Volcanic plume model

Define
mass flux $\quad Q = \rho U b^2$, $\qquad$ momentum flux $\quad M = \rho U b^2$,
column temperature $\quad T$, $\qquad$ mass fraction of gas $\quad n$,
and mass fraction of solids $\quad 1 - n$,
and formulate conservation equations through control volumes:

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2U_e \rho_e b, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = g\left(\rho_e - \rho\right) b^2, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\Big(\left(1-n\right) Q\Big) = 0$$

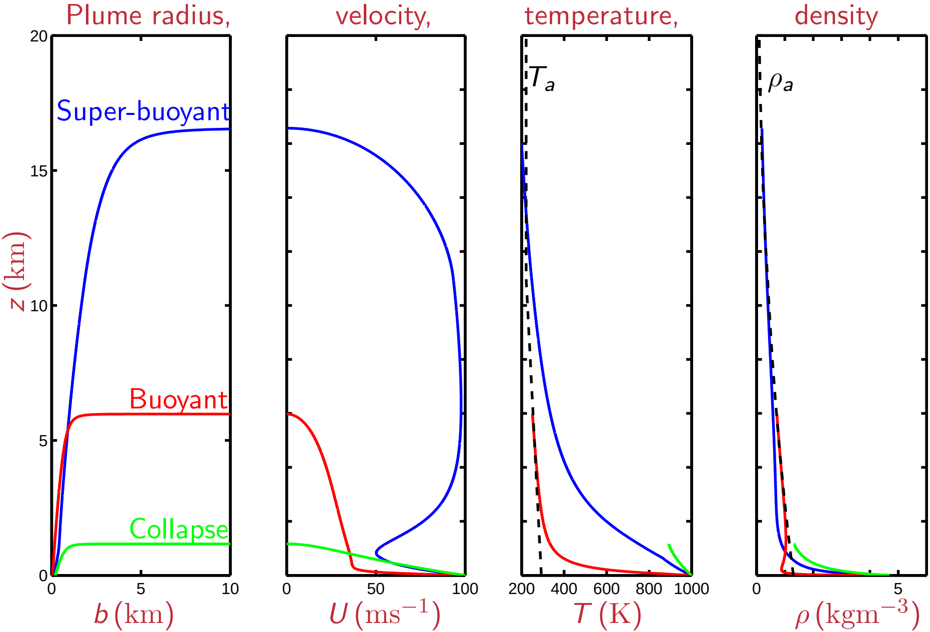
$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(\left(C_p T + gz + \tfrac{1}{2}U^2\right) Q\Big) = 2U_e \rho_e b \left(C_a T_a + gz\right).$$

Constitutive equations:

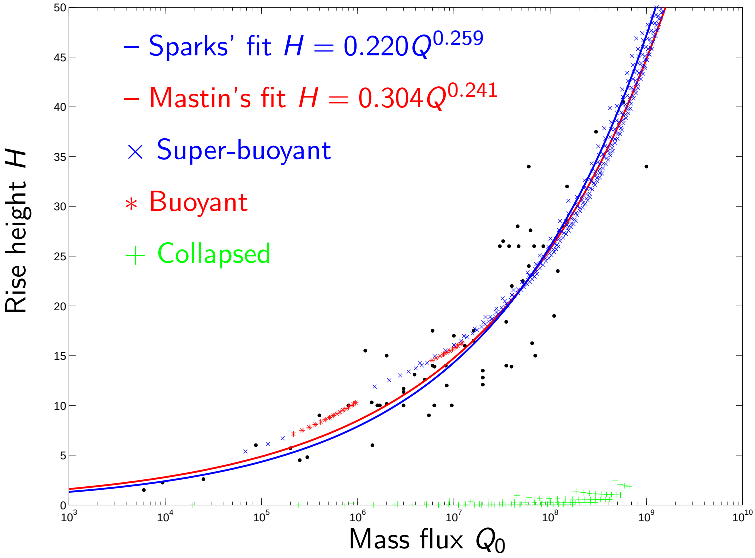
$$\frac{1}{\rho} = \frac{1-n}{\rho_s} + \frac{n R_g T}{P_a},$$

$$C_p = \frac{Q_a}{Q} C_a + \frac{Q_m}{Q} C_m + \frac{Q_s}{Q} C_s, \qquad R_g = \frac{Q_a}{Q_g} R_a + \frac{Q_m}{Q_g} R_m$$

Volcanic model solutions



Comparison to volcanic plume observations



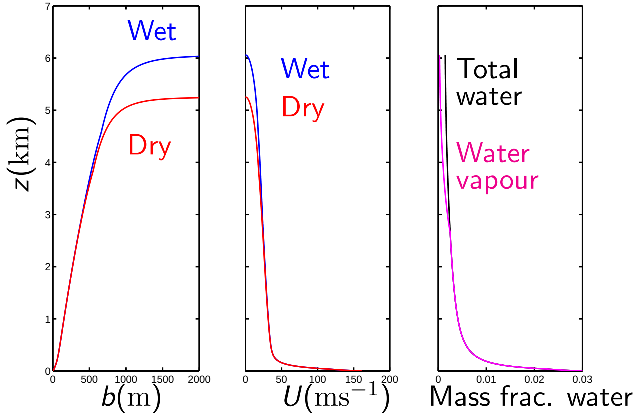
3. Atmospheric controls

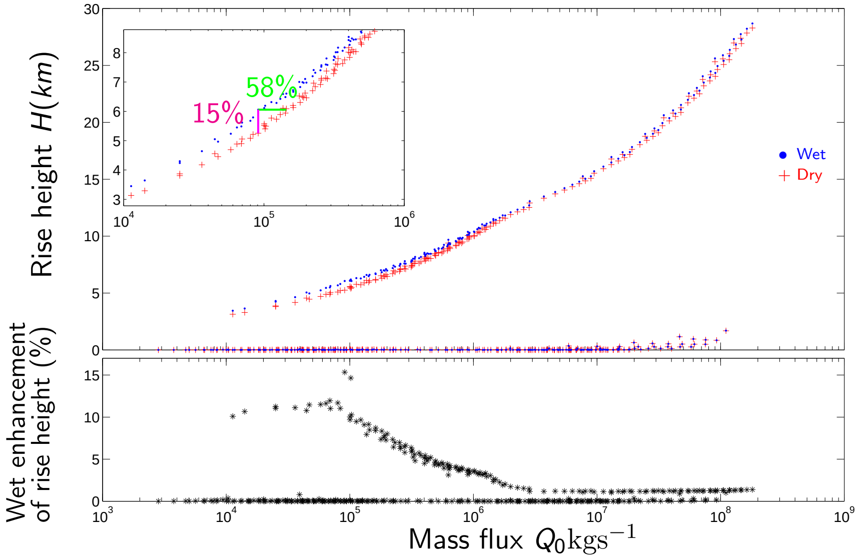
Wet plumes & wind-blown plumes

Eyjafjallajökull 20 April 2010 (Vodaphone)

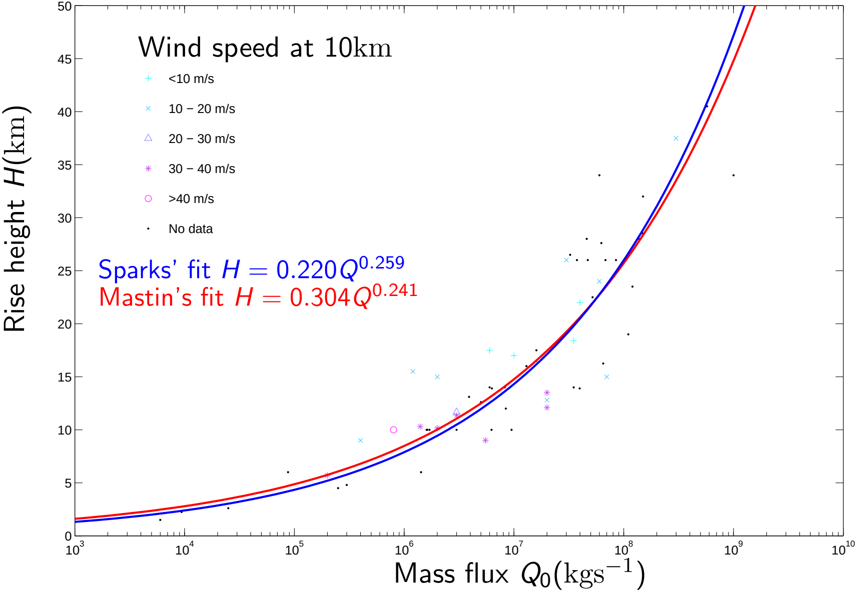
Wet plumes

Water vapour in the plume can condense as it is raised to higher altitude, releasing latent heat.
Water added at the vent or through entrainment of moist atmospheric air has been included in the volcanic plume model.




Rise height $H(km)$
Wet enhancement of rise height (%)
Mass flux $Q_0 \mathrm{kgs}^{-1}$
Wet
Dry
15%
58%
30
25
20
15
10
5
0
8
7
6
5
4
3
10^4
10^5
10^6
10^3
10^7
10^8
10^9

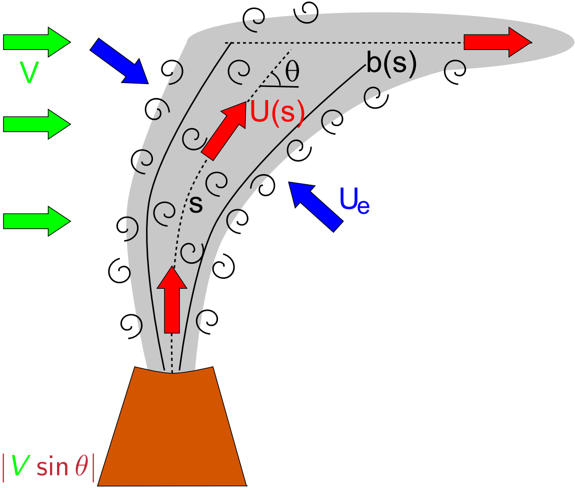
Wind-blown plumes – observations



Wind-blown plumes

Hewett, Fay & Hoult (1971) model of buoyant plume in cross-wind.

Formulate conservation equations in a plume-centered coordinate system.

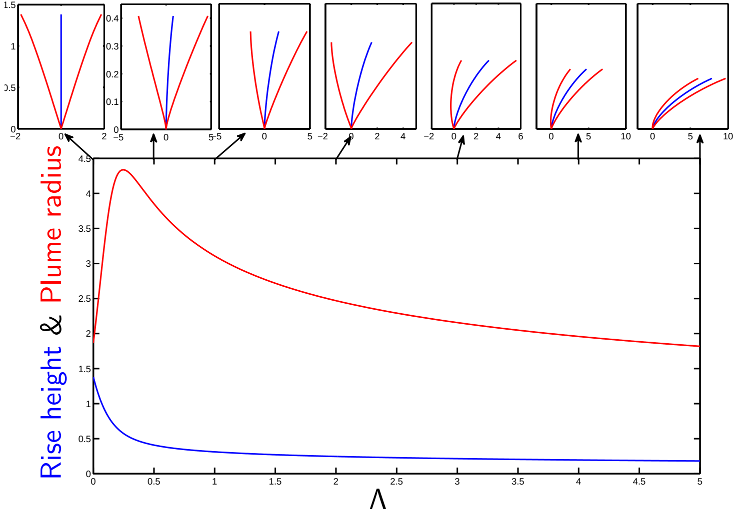


Air entrained into plume has horizontal momentum.

Entrainment velocity into the plume is

$U_e = k_s \left| U(s) - V\cos\theta \right| + k_w \left| V \sin\theta \right|$

Dimensionless controlling parameters: $\gamma = \frac{k_w}{k_s} \approx 10, \qquad \Lambda = \frac{V k_s^{1/2}}{F_0^{1/4} N^{1/4}}.$



Conclusions

- Simple mathematical models provide a scaling relationship between rise height and mass flux $H \sim Q_0^{1/4}$.
- Observations broadly follow this scaling, but some significant scatter.
- Volcanic plume models allow additional physics to be modelled, and recover $H \sim Q_0^{1/4}$ scaling.
- Including water phase change in plume model suggests moisture content of atmosphere can result in significant overestimation of source flux based on rise height.
- Wind may limit rise height of plumes from small/moderately sized eruptions.
- A simple mathematical model of wind-blown plumes can be formulated.