Mathematical models of volcanic plumes

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1. Modelling approaches

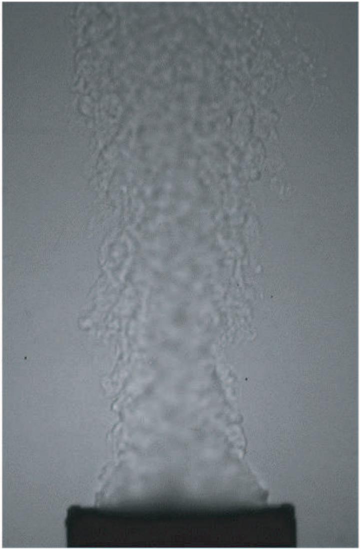
Mount Augustine, 1986 (USGS)

Turbulent buoyant plumes

Density difference between plume and environment $\rightarrow$ **bulk vertical motion**

Typically high Reynolds number and high Rayleigh number $\rightarrow$ **turbulent motion**

Turblent eddies engulf parcels of ambient fluid and mix it with plume fluid.

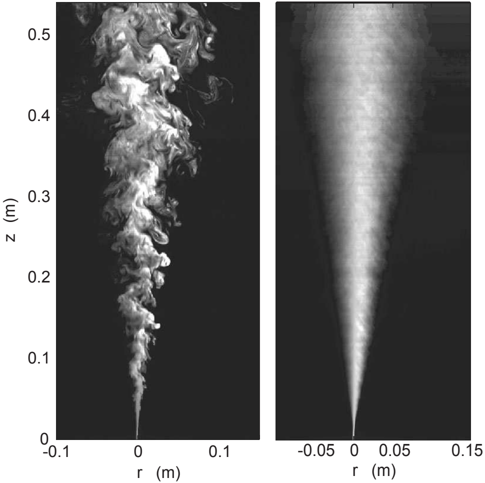


Hunt, G.R. & van den Bremer (2011) Classical plume theory: 1937–2010 and beyond. *IMA J. Appl. Math.* **76**, 424–448

Modelling approaches

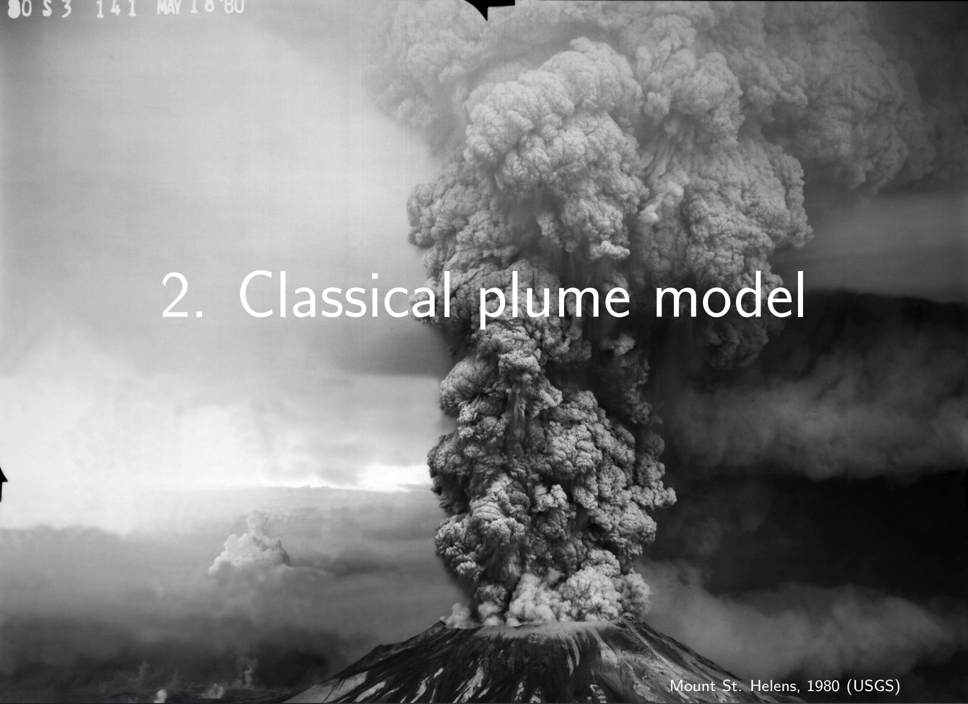
Numerical solution of Navier-Stokes equations – must resolve or parameterize turbulent eddies on a wide range of scales.

Integral model describing variation of averaged plume quantities – look on a time scale longer than eddy turnover time and make assumptions of plume structure and entrainment of ambient fluid.



Scase, M.M., Caulfield, C.P. & Dalziel, S.B. (2008) *J. Fluid Mech.* **600**, 181–199

2. Classical plume model

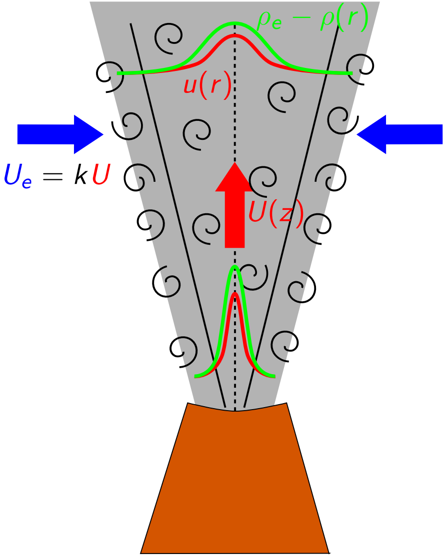


Mount St. Helens, 1980 (USGS)

Morton, Taylor, Turner (1956) model of buoyant convection from maintained source

Key assumptions:

- self-similarity of mean profiles;
- linear dependence of density on concentration;
- entrainment velocity proportional to a characteristic velocity of the plume;



Morton, Taylor, Turner (1956) model of buoyant convection from maintained source

Replacing velocity and buoyancy profiles with equivalent top-hat profiles and defining:

mass flux

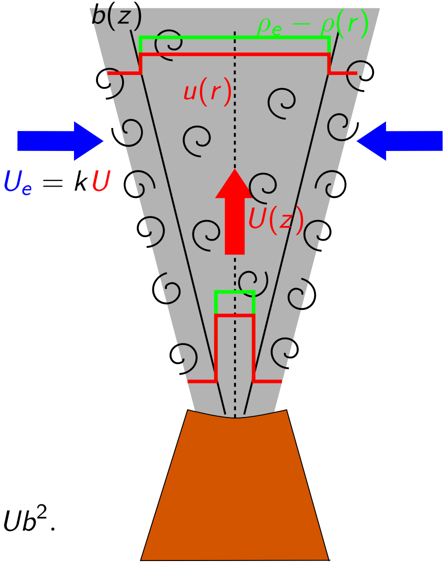
$$\pi Q = 2\pi \int \rho u r \mathrm{d}r = \pi \rho U b^2,$$

momentum flux

$$\pi M = 2\pi \int \rho u^2 r \mathrm{d}r = \pi \rho U^2 b^2,$$

buoyancy flux

$$\pi F = 2\pi \int g(\rho_e - \rho) u r \mathrm{d}r = \pi g (\rho_e - \rho) U b^2.$$



Morton, Taylor, Turner (1956) model of buoyant convection from maintained source

Conservation of mass flux, momentum flux and buoyancy flux through a control volume leads to a system of 3 governing equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}(b^2U) = 2kUb, \qquad \frac{\mathrm{d}Q}{\mathrm{d}z} = 2k\sqrt{\rho_0 M},$$

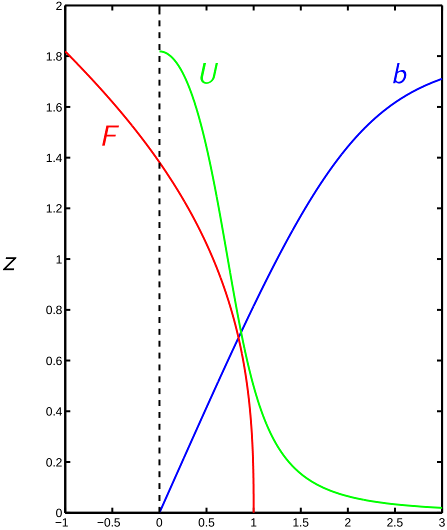
$$\frac{\mathrm{d}}{\mathrm{d}z}(b^2U^2) = b^2 g\frac{\rho_e - \rho}{\rho_0}, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{FQ}{M},$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(b^2 Ug\frac{\rho_e - \rho}{\rho_0}\right) = b^2 U\frac{g}{\rho_0}\frac{\mathrm{d}\rho_e}{\mathrm{d}z}, \qquad \frac{\mathrm{d}F}{\mathrm{d}z} = -N^2 Q,$$

with ambient stratification represented by buoyancy frequency N with

$$N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_e}{\mathrm{d}z}.$$

Uniform stratification $N^2 = \mathrm{const.}$



Uniform stratification $N^2 = \mathrm{const.}$

For a pure plume there are 3 dimensional parameters: the source buoyancy flux F_0, the buoyancy frequency N and a density scale ρ_0. Therefore the rise height, H, scales as

$$H \sim (F_0/\rho_0)^{1/4} N^{-3/4}.$$

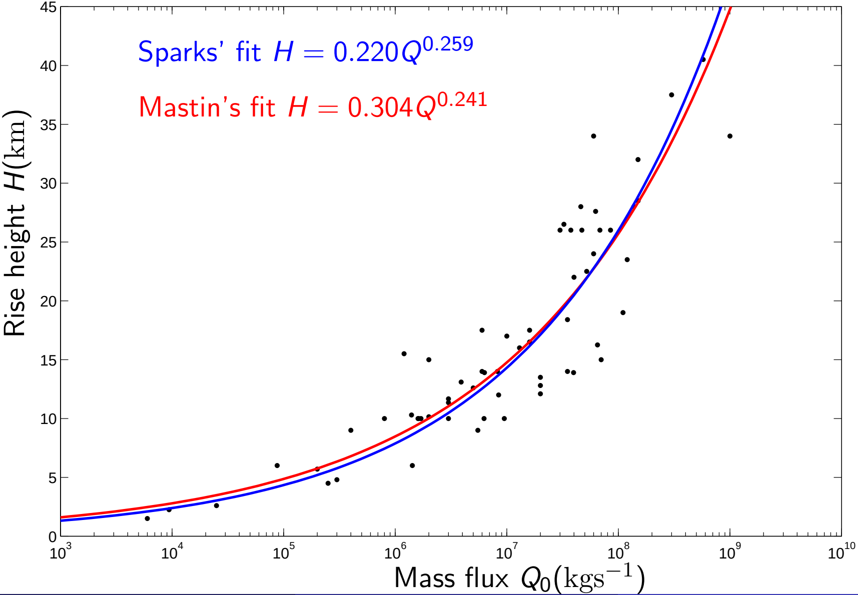
An estimate for rise height of volcanic plumes can be made by relating F_0 to source mass flux, Q_0,

$$\frac{F_0}{\rho_0} = \frac{g}{\rho_0}\left(\frac{\rho_e - \rho}{\rho_0}\right) Q_0 \approx \frac{g}{\rho_0}\left(\frac{T - T_a}{T_a}\right) Q_0,$$

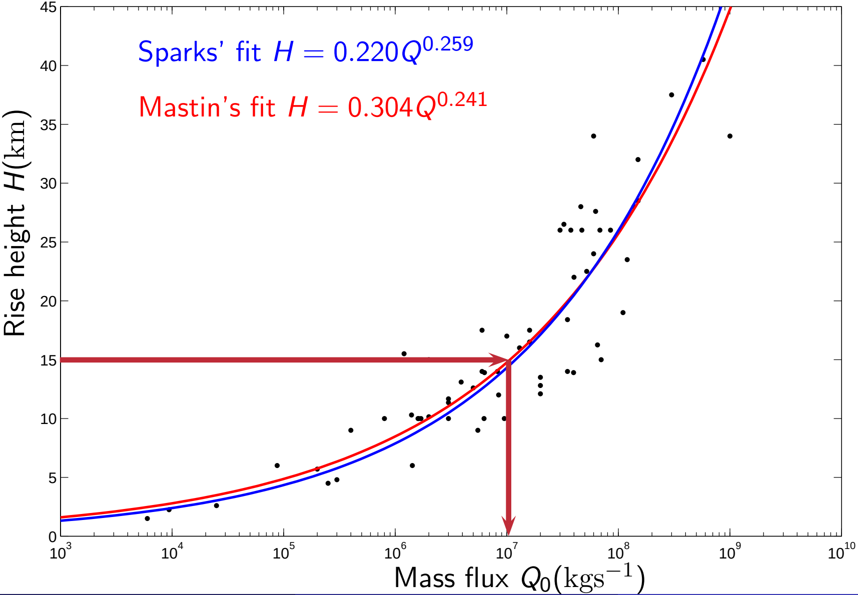
so

$$H \sim Q_0^{1/4} N^{-3/4}.$$

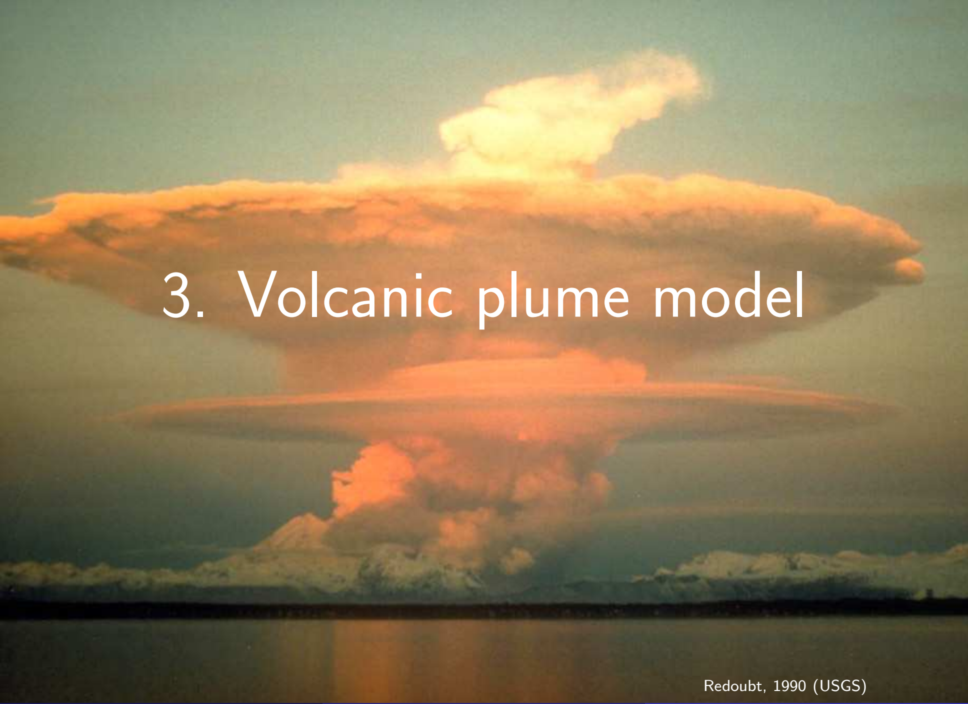
Comparison to volcanic plume observations



Comparison to volcanic plume observations

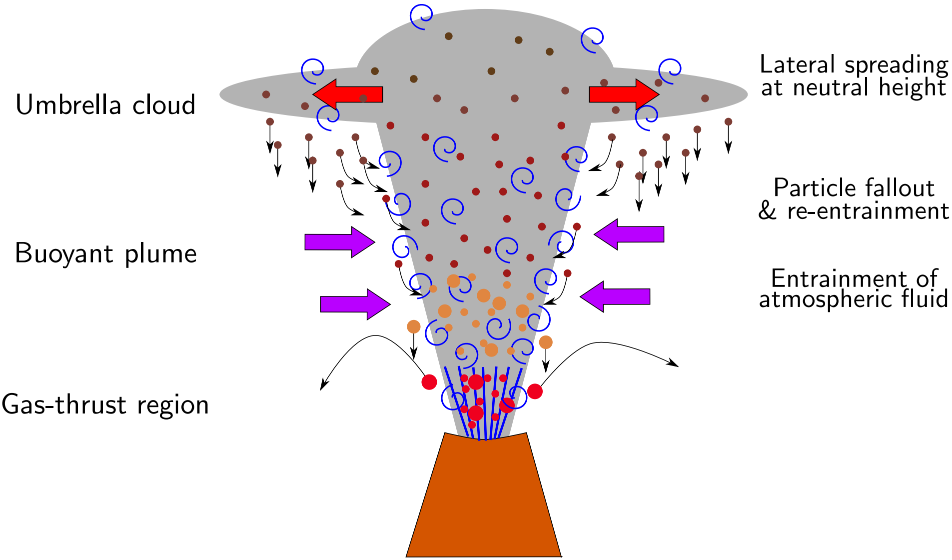


3. Volcanic plume model



Redoubt, 1990 (USGS)

Volcanic plume structure



Volcanic plume model

Woods (1988) developed a model of volcanic plumes, extending the classical Morton, Taylor, Turner model with descriptions of:

- the heat content of the erupting material
- heat enchange between particles and entrained air
- the thermodynamic expansion of the gases
- nonlinear variations of density with temperature
- varying atmospheric stratification
- different entrainment rates in jet and plume regions

Volcanic plume model

Define
mass flux $\quad Q = \rho U b^2,$ $\qquad$ momentum flux $\quad M = \rho U b^2,$
column temperature $\quad T,$ $\qquad$ mass fraction of gas $\quad n,$
and mass fraction of solids $\quad 1 - n,$
and formulate conservation equations through control volumes:

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2U_e \rho_e b, \qquad \frac{\mathrm{d}M}{\mathrm{d}z} = g\left(\rho_e - \rho\right) b^2, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\Big(\left(1 - n\right) Q\Big) = 0$$

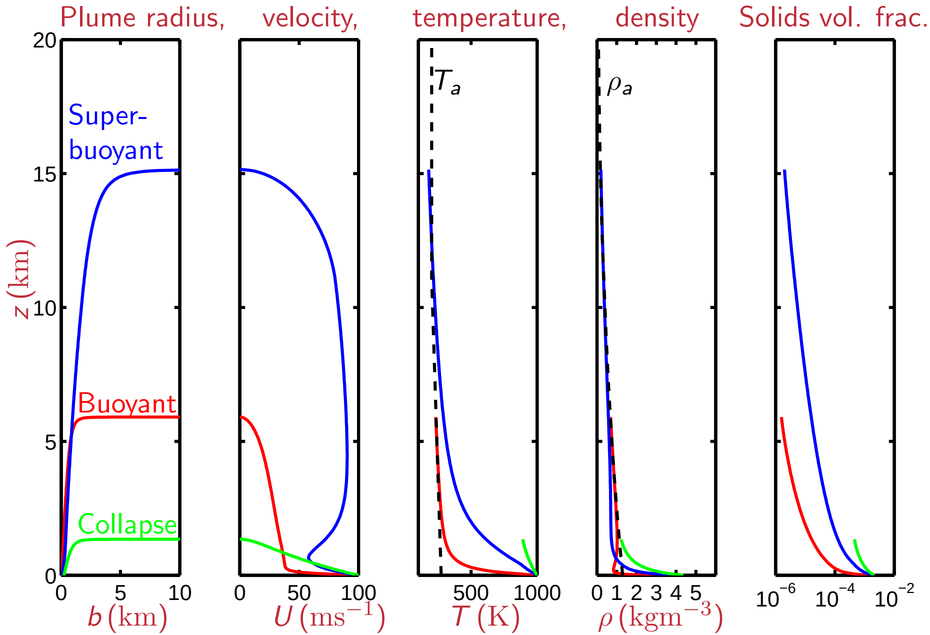
$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(\left(C_p T + gz + \tfrac{1}{2}U^2\right) Q\Big) = 2U_e \rho_e b \left(C_a T_a + gz\right).$$

Constitutive equations:

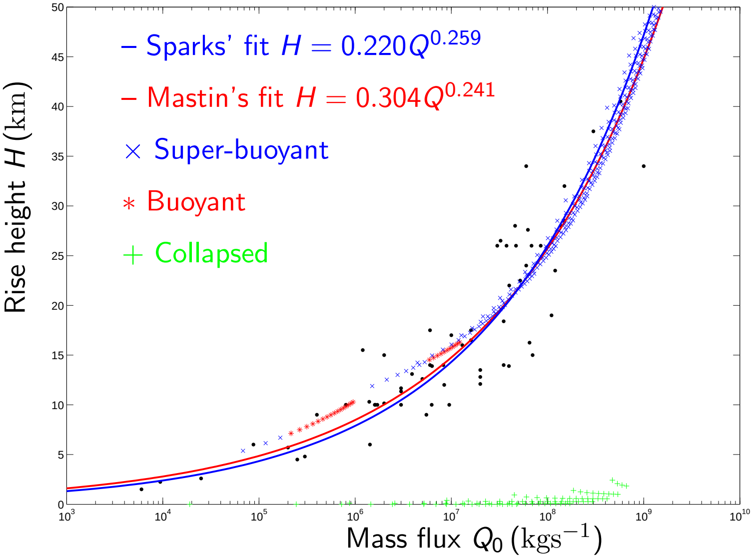
$$\frac{1}{\rho} = \frac{1 - n}{\rho_s} + \frac{n R_g T}{P_a},$$

$$C_p = \frac{Q_a}{Q} C_a + \frac{Q_m}{Q} C_m + \frac{Q_s}{Q} C_s, \qquad R_g = \frac{Q_a}{Q_g} R_a + \frac{Q_m}{Q_g} R_m$$

Volcanic model solutions



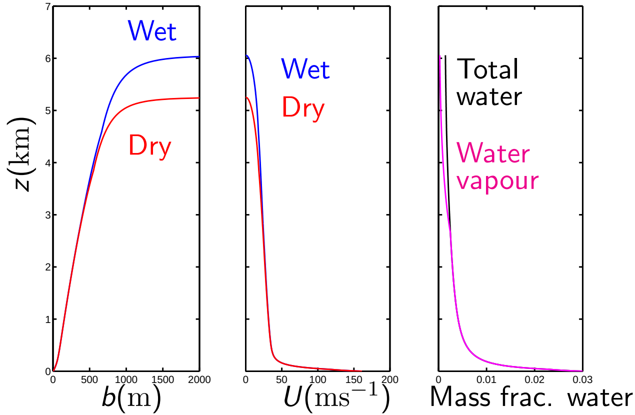
Comparison to volcanic plume observations



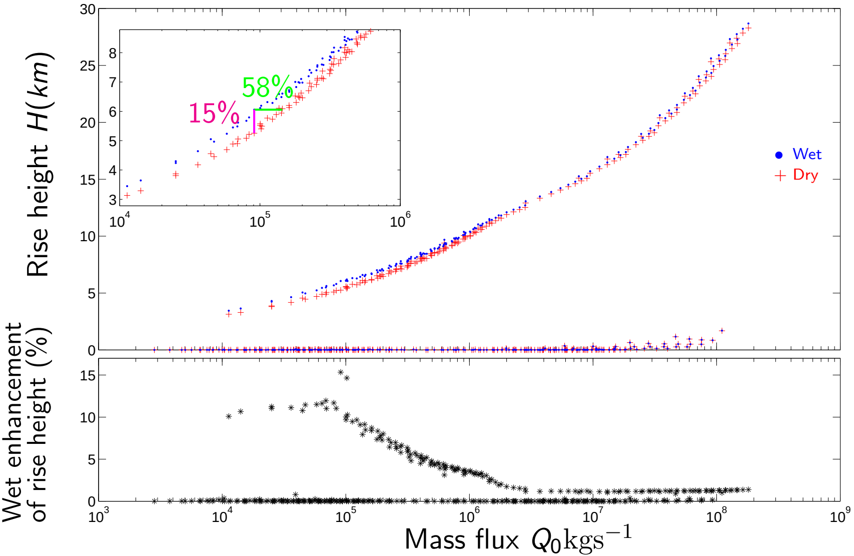
Wet plumes

Water vapour in the plume can condense as it is raised to higher altitude, releasing latent heat.

Water added at the vent or through entrainment of moist atmospheric air has been included in the volcanic plume model.



Wet plumes

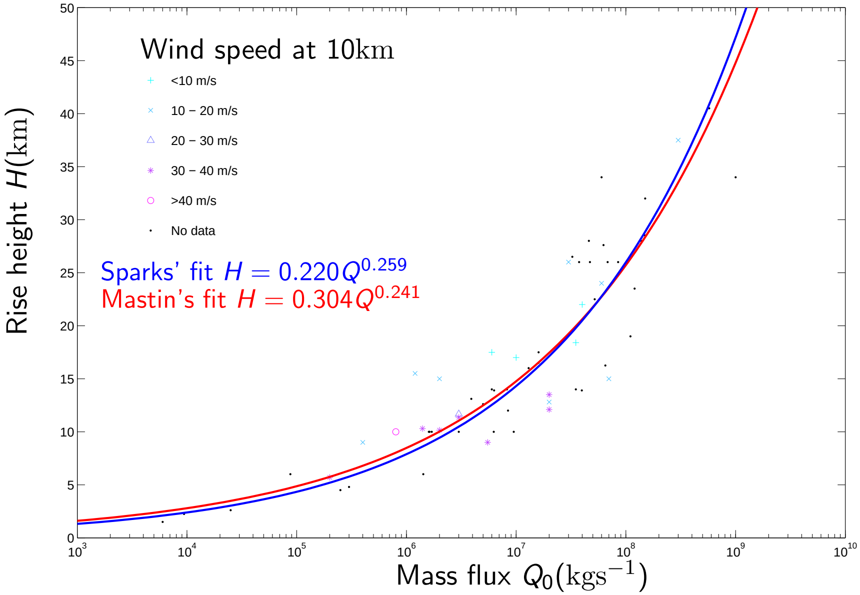


Rise height $H(km)$
Wet enhancement of rise height (%)
Mass flux $Q_0 \mathrm{kgs}^{-1}$
Wet
Dry
58%
15%
30
25
20
15
10
5
0
8
7
6
5
4
3
10^4
10^5
10^6
10^3
10^7
10^8
10^9

4. Wind-blown plumes

Eyjafjallajökull 20 April 2010 (Vodaphone)

Wind-blown plumes – observations



Wind-blown plumes

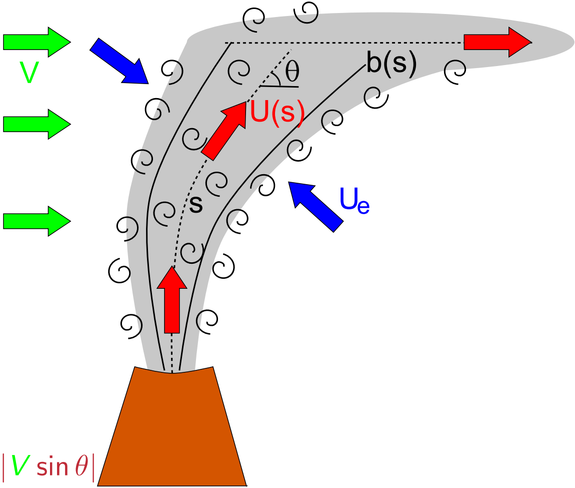
Hewett, Fay & Hoult (1971) model of buoyant plume in cross-wind.

Formulate conservation equations in a plume-centered coordinate system.

Air entrained into plume has horizontal momentum.

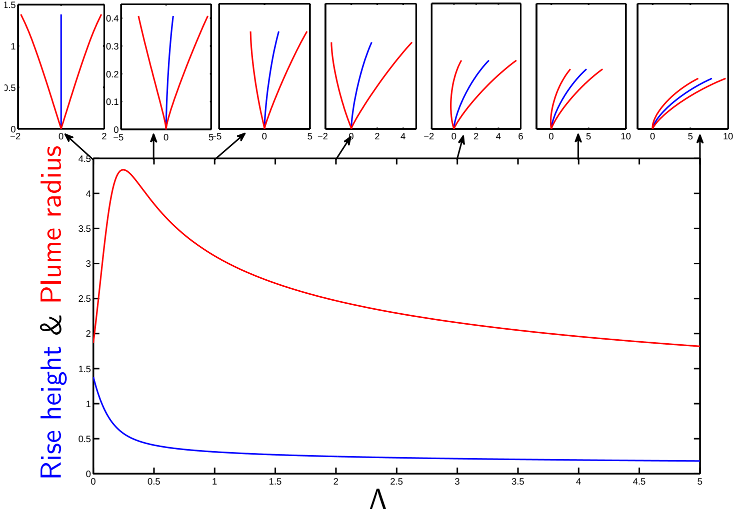
Entrainment velocity into the plume is

$$U_e = k_s \left| U(s) - V \cos\theta \right| + k_w \left| V \sin\theta \right|$$



Bent-over plume model – uniform stratification

Dimensionless controlling parameters: $\gamma = \frac{k_w}{k_s} \approx 10, \qquad \Lambda = \frac{V k_s^{1/2}}{F_0^{1/4} N^{1/4}}.$



5. Time-dependent plumes

Puyehue 2011 (National Geographic)

Time varying sources

Time-dependence has been included into the classical plume model to describe the response of a plume to changes in the source conditions (Scase, Caulfield, Dalziel & Hunt 2006).

The time-dependent governing equations are

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho b^2\right) + \frac{\partial}{\partial z}\left(\rho b^2 U\right) = 2k\sqrt{\rho_0 M},\\
&\frac{\partial}{\partial t}\left(\rho b^2 U\right) + \frac{\partial}{\partial z}\left(\rho b^2 U^2\right) = b^2 g\left(\rho_e - \rho\right),\\
&\frac{\partial}{\partial t}\left(\left(\rho_e - \rho\right) g b^2\right) + \frac{\partial}{\partial z}\left(\left(\rho_e - \rho\right) g b^2 U\right) = -N^2 \rho_e b^2 U.
\end{aligned}$$

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The time-dependent governing equations are

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{Q^2}{M}\right) + \frac{\partial Q}{\partial z} = 2k\sqrt{\rho_0 M},\\
&\frac{\partial Q}{\partial t} + \frac{\partial M}{\partial z} = \frac{QF}{M},\\
&\frac{\partial}{\partial t}\left(\frac{QF}{M}\right) + \frac{\partial F}{\partial z} = -N^2\left(Q + \frac{F}{g}\right).
\end{aligned}
$$

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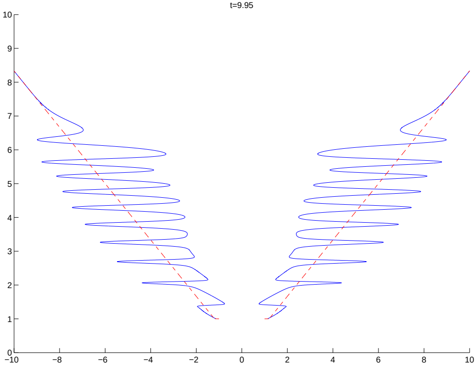

Conclusions

- Integral models provide insight into essential physical processes and controlling parameters.
- Simple mathematical models provide a scaling relationship between rise height and mass flux $H \sim Q_0^{1/4}$.
- Volcanic plume models allow additional physics to be modelled, and recover $H \sim Q_0^{1/4}$ scaling in calm environments.
- Wind may limit rise height of plumes from small/moderately sized eruptions.
- Time variation of source conditions can be included into integral model.